



Research Article

Hermite–Hadamard-Type Inequalities for Generalized Preinvex Functions

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Abstract

In this research work, we introduce the concept of an s -type n -polynomial m -exponential preinvex function and establish several fundamental results concerning its algebraic properties. Furthermore, by employing this newly defined class of functions, we derive improved forms of the Hermite–Hadamard (H–H) inequality and obtain several refinements of H–H-type inequalities. The results established in this study provide meaningful extensions and generalizations of several existing results in the literature. As special cases, these findings are further illustrated through appropriate corollaries, which recover and extend a number of well-known inequalities.

1. Introduction

Inequality theory and convex analysis form important pillars of contemporary mathematical analysis, with broad applications across both pure and applied mathematics. Among these concepts, convexity provides a powerful framework for studying the structural properties of functions and plays a central role in diverse areas, including optimization, functional analysis, numerical analysis, and mathematical modeling. A tremendous book written by Hardy *et.al* [1], describes the detailed concepts of convexity of any function. Some useful and significant applications of different convexities are discussed in [2, 3, 4, 5], that resolve many problems of fractional inequalities.

The invex function is defined by Hanson [6]. The concept of invex and preinvex function was given by Ben-Israel and Mond [7]. The theory of convexity performed a vital role in distinct areas like information theory, coding theory, engineering, optimization and inequality theory. The generalizations, extensions, variants, and refinements for different type of convexity are considered by many scientist. Toader [8] firstly explored namely m -convex functions. Later on, Latif and Shoaib [9] and Deng *et.al* [10] discussed the generalized idea of preinvexity by utilizing m -preinvex function. Toplu *et al.* [11] developed a generalized form of the Hermite–Hadamard-type inequality and examined the framework of n -polynomial convexity. Subsequently, Rashid *et al.* [12] introduced the concept of n -polynomial s -type convex functions and established several associated integral inequalities, thereby extending the existing results in the theory of convexity and inequalities.

Definition 1.1. [13] Let $U \subseteq \mathbb{R}^n$ be a nonempty set. We say that U is ***invex with respect to*** the bifunction $\xi(\cdot, \cdot)$ if, for every $\Theta, \mu_2 \in U$ and $w \in [0, 1]$, the point $\mu_2 + w \xi(\Theta, \mu_2)$ belongs to U .

$$\Theta + w \xi(\Theta, \mu_2) \in U, \quad (1)$$

the U set mentioned above is called also the ξ –connected set.

Definition 1.2. [14] Let $U \subseteq \mathbb{R}^n$ and consider the mapping $\xi : U \times U \times (0, 1] \longrightarrow \mathbb{R}^n$. The set U is said to be *** m -invex with respect to ξ *** if, for every $\Theta, \mu_2 \in U$, $m \in (0, 1]$, and $w \in [0, 1]$, the following inclusion holds: $m\mu_2 + w \xi(\Theta, \mu_2, m) \in U$. holds for every $\Theta, \mu_2 \in U$, $\theta \in [0, 1]$, and $m \in (0, 1]$.

Definition 1.3. [13] The φ function on the U invex set with respect to ξ is noted to be pre-invex, if

$$\varphi(\Theta + w \xi(\Theta, \mu_2)) \leq w \varphi(\Theta) + (1 - w) \varphi(\mu_2), \quad \forall w \in [0, 1], \Theta, \mu_2 \in U. \quad (2)$$

The $-\varphi$ function is noted to be preincave if and only if φ is preinvex. The Mohan and Neogy[15] function ξ discussed the hypothesis regarding ξ is mention below.

Condition for C : Condition C plays a significant role in the development and analysis of various types of invexity and related inequalities. This condition was introduced and studied in the context of invex sets by Noor [16] and Farajzadeh *et al.* [17].

Let $U \subseteq \mathbb{R}^n$ be an open invex set with respect to the mapping

$$\xi : U \times U \longrightarrow \mathbb{R}^n.$$

Then, for any $\Theta, \mu_2 \in U$ and $w \in [0, 1]$, the mapping ξ is said to satisfy Condition C if

$$\xi(\mu_2, \mu_2 + w \xi(\Theta, \mu_2)) = -w \xi(\Theta, \mu_2),$$

and

$$\xi(\Theta, \mu_2 + w\xi(\Theta, \mu_2)) = (1 - w)\xi(\Theta, \mu_2).$$

$$\xi(\mu_2, \mu_2 + w\xi(\Theta, \mu_2)) = -w\xi(\Theta, \mu_2) \quad (3)$$

$$\xi(\Theta, \mu_2 + w\xi(\Theta, \mu_2)) = (1 - w)\xi(\Theta, \mu_2), \quad (4)$$

for any $w_1, w_2 \in [0, 1]$ and $\Theta, \mu_2 \in U$ from condition C, we have

$$\xi(\mu_2 + w_2\xi(\Theta, \mu_2), \mu_2 + w_1\xi(\Theta, \mu_2)) = (w_2 - w_1)\xi(\Theta, \mu_2). \quad (5)$$

Extended condition C: In terms of m -preinvexity, the Ting Song Du [18] extended the C condition. Let $U \subset \mathbb{R}^n$ with respect to $\xi : U \times U \times (0, 1] \rightarrow \mathbb{R}^n$, be an open m -invex subset for any $w \in [0, 1]$, and $\Theta, \mu_2 \in U$, then

$$\xi(\mu_2, m\mu_2 + w\xi(\Theta, \mu_2, m)) = -w\xi(\Theta, \mu_2, m) \quad (6)$$

$$\xi(\Theta, m\mu_2 + w\xi(m\Theta, \mu_2, m)) = (1 - w)\xi(\Theta, \mu_2, m) \quad (7)$$

$$\xi(\Theta, \mu_2, m) = -\xi(\mu_2, \Theta, m). \quad (8)$$

Definition 1.4. [19] If $\varphi : \mathbb{U} \rightarrow \mathbb{R}$, then φ with respect to $\xi : \mathbb{U} \times \mathbb{U} \times (0, 1] \rightarrow \mathbb{R}^n$, is said to be generalized m -pre-invex for fixed $m \in (0, 1]$, if

$$\varphi(m\mu_2 + w\xi(\Theta, \mu_2, m)) \leq w\varphi(\Theta) + m(1 - w)\varphi(\mu_2), \quad (9)$$

holds for every $w \in [0, 1]$, $\Theta, \mu_2 \in \mathbb{U}$.

Definition 1.5. [12] A $\varphi : \mathbb{U} \rightarrow \mathbb{R}$ non-negative function is noted to be s -type convex function if $\forall \Theta, \mu_2 \in \mathbb{U}, n \in \mathbb{N}, w, s \in [0, 1]$, if

$$\varphi(w\Theta + (1 - w)\mu_2) \leq [1 - (s(1 - w))]^n \varphi(\Theta) + [1 - sw]^n \varphi(\mu_2). \quad (10)$$

Definition 1.6. [11] A $\varphi : \mathbb{U} \rightarrow \mathbb{R}$ real valued non-negative function is noted to be n -polynomial convex function, if

$$\varphi(w\Theta + (1 - w)\mu_2) \leq \frac{\sum_{i=1}^n [1 - (1 - w)^i] \varphi(\Theta)}{n} + \frac{\sum_{i=1}^n [1 - w^i] \varphi(\mu_2)}{n}, \quad (11)$$

$\forall \Theta, \mu_2 \in \mathbb{U}, n \in \mathbb{N}, s \in [0, 1]$ and $w \in [0, 1]$.

Definition 1.7. [12] A nonnegative function $\varphi : I \rightarrow \mathbb{R}$ is said to be an s -type n -polynomial convex function on I if, for every $\Theta, \mu_2 \in I, n \in \mathbb{N}$, and $s, w \in [0, 1]$, it satisfies

$$\varphi(w\Theta + (1 - w)\mu_2) \leq \frac{1}{n} \sum_{k=1}^n [(1 - (1 - w)^k)^s \varphi(\Theta) + (1 - w^k)^s \varphi(\mu_2)].$$

$$\varphi(w\Theta + (1 - w)\mu_2) \leq \frac{\sum_{i=1}^n [1 - (s(1 - w))^i] \varphi(\Theta)}{n} + \frac{\sum_{i=1}^n [1 - (sw)^i] \varphi(\mu_2)}{n}. \quad (12)$$

Definition 1.8. [19] Consider a nonempty m -invex set $\mathbb{U} \subset \mathbb{R}$ with respect to $\xi : \mathbb{U} \times \mathbb{U} \times (0, 1] \rightarrow \mathbb{R}$, then n -polynomial s -type m -pre-invex for $\varphi : U \rightarrow \mathbb{R}$ as,

$$\varphi(m\mu_2 + w\xi(\Theta, \mu_2, m)) \leq \frac{\sum_{i=1}^n [1 - (s(1-w))^i]}{n} \varphi(\Theta) + \frac{\sum_{i=1}^n m^i [1 - (sw)^i]}{n} \varphi(\mu_2), \quad (13)$$

holds for every $\Theta, \mu_2 \in I$, $s, w \in [0, 1]$, and $n \in \mathbb{N}$.

Definition 1.9. [20] Two functions are similarly ordered functions if

$$(\varphi(\Theta) - \varphi(\mu_2))(\phi(\Theta) - \phi(\mu_2)) \geq 0, \quad \forall \Theta, \mu_2 \in \mathfrak{R} \quad (14)$$

Definition 1.10. [21] Let a nonempty invex set $U \in \mathbb{R}$ with respect to $\xi : U \times U \rightarrow \mathbb{R}$, is said to be s -type n -polynomial pre-invex with the function $\varphi : U \rightarrow \mathbb{R}$, if

$$\varphi(\mu_2 + w\xi(\Theta, \mu_2)) \leq \frac{\sum_{i=1}^n [1 - (s(1-w))^i]}{n} \varphi(\Theta) + \frac{\sum_{i=1}^n [1 - (sw)^i]}{n} \varphi(\mu_2), \quad (15)$$

holds for $\Theta, \mu_2 \in U$, $s, w \in [0, 1]$, $w \in [0, 1]$, $n \in \mathbb{N}$.

Definition 1.11. An invex set U and real valued φ function is called exponentially pre-invex, if the inequality holds for $\theta \in \mathbb{R}$

$$\varphi(\Theta + w\xi(\Theta, \mu_2)) \leq (1-w) \frac{\varphi(\Theta)}{e^{\theta\Theta}} + w \frac{\varphi(\mu_2)}{e^{\theta\mu_2}}, \quad (16)$$

exponentially pre-invex (ep) function defined by Mubeen et.al in [22] definition (2.8) and mention its limitations.

2. m -Exponentially Pre-invex (m-EP)

Definition 2.1. If $\varphi : \mathbb{U} \rightarrow \mathbb{R}$, with respect to $\xi : \mathbb{U} \times \mathbb{U} \times (0, 1] \rightarrow \mathbb{R}^n$, then φ is noted to be m -exponentially pre-invex for $\theta \in \mathfrak{R}$, and some fixed $m \in (0, 1]$, if

$$\varphi(m\mu_2 + w\xi(\Theta, \mu_2, m)) \leq w \frac{\varphi(\Theta)}{e^{\theta\Theta}} + m(1-w) \frac{\varphi(\mu_2)}{e^{\theta\mu_2}} \quad (17)$$

holds for every $\Theta, \mu_2 \in \mathbb{U}$, $w \in [0, 1]$.

If we take $\theta = 0$, we get generalized m -pre-invex [19]

If we replace $s = 1$ in [22], definition 2.10, we have (17).

3. s -type n -polynomial m -exponentially pre-invex

In this section, we introduce s -type n -polynomial m -exponentially pre-invex function and also discuss its some algebraic applications.

Definition 3.1. A non empty m -invex set $U \subset \mathbb{R}$, with respect to $\xi : U \times U \times (0, 1] \rightarrow \mathbb{R}$ is called s -type n -polynomial m -exponentially pre-invex for the function $\varphi : U \rightarrow \mathbb{R}$, if

$$\varphi(m\mu_2 + w\xi(\Theta, \mu_2, m)) \leq \frac{\sum_{i=1}^n (1 - (s(1-w))^i)}{n} \frac{\varphi(\Theta)}{e^{\theta\Theta}} + \frac{\sum_{i=1}^m (1 - (sw)^i) m^i}{n} \frac{\varphi(\mu_2)}{e^{\theta\mu_2}}, \quad (18)$$

holds for every $\Theta, \mu_2 \in U$, $s \in [0, 1]$, $m \in (0, 1]$, $w \in [0, 1]$, $n \in \mathbb{N}$, $\theta \in \mathfrak{R}$.

If we take $\theta = 0$, the s -type n -polynomial m -exponentially pre-invex function reduces to s -type n -polynomial m -pre-invex function defined in [19]

Theorem 3.2. Let $\varphi, \phi : I = [\ominus, \mu_2] \rightarrow \mathbb{R}$ be two s -type n -polynomial m -exponentially pre-invex functions. Then the following properties are satisfied:

1. The function $\varphi + \phi$ is also an s -type n -polynomial m -exponentially pre-invex function.
2. For any constant $c \in \mathbb{R}$ with $c \geq 0$, the scalar multiple $c\varphi$ is an s -type n -polynomial m -exponentially pre-invex function.

Proof. 1. $\varphi + \phi$ is s -type n -polynomial m -exponentially pre-invex function.

$$\begin{aligned} & (\varphi + \phi)(m\mu_2 + w\xi(\ominus, \mu_2, m)) = \\ & (\varphi)(m\mu_2 + w\xi(\ominus, \mu_2, m)) + (\phi)(m\mu_2 + w\xi(\ominus, \mu_2, m)) \\ & \leq \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{\varphi(\ominus)}{e^{\theta\ominus}} + \frac{1}{n} \sum_{i=1}^m m^i (1 - (sw)^i) \frac{\varphi(\mu_2)}{e^{\theta\mu_2}} \\ & + \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{\phi(\ominus)}{e^{\theta\ominus}} + \frac{1}{n} \sum_{i=1}^m (1 - (sw)^i) m^i \frac{\phi(\mu_2)}{e^{\theta\mu_2}} \\ & = \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{\varphi(\ominus) + \phi(\ominus)}{e^{\theta\ominus}} + \frac{1}{n} \sum_{i=1}^m m^i (1 - (sw)^i) \frac{\varphi(\mu_2) + \phi(\mu_2)}{e^{\theta\mu_2}} \\ & = \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{(\varphi + \phi)(\ominus)}{e^{\theta\ominus}} + \frac{1}{n} \sum_{i=1}^m m^i (1 - (sw)^i) \frac{(\varphi + \phi)(\mu_2)}{e^{\theta\mu_2}} \quad (19) \end{aligned}$$

hence proved.

2. $(c\varphi)$ is s -type n -polynomial m -exponentially pre-invex function.

$$\begin{aligned} & (c\varphi)(m\mu_2 + w\xi(\ominus, \mu_2, m)) \\ & \leq c \left[\frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{\varphi(\ominus)}{e^{\theta\ominus}} + \frac{1}{n} \sum_{i=1}^m m^i (1 - (sw)^i) \frac{\varphi(\mu_2)}{e^{\theta\mu_2}} \right] \\ & = \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{c\varphi(\ominus)}{e^{\theta\ominus}} + \frac{1}{n} \sum_{i=1}^m m^i (1 - (sw)^i) \frac{c\varphi(\mu_2)}{e^{\theta\mu_2}} \\ & = \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{(c\varphi)(\ominus)}{e^{\theta\ominus}} + \frac{1}{n} \sum_{i=1}^m m^i (1 - (sw)^i) \frac{(c\varphi)(\mu_2)}{e^{\theta\mu_2}} \quad (20) \end{aligned}$$

hence proved. □

Theorem 3.3. Let $\varphi : I \rightarrow L$ is s -type n -polynomial m -exponentially pre-invex function with respect to ξ and $\phi : L \rightarrow \mathbb{R}$ is non decreasing function. Then $(\varphi \circ \phi)$ is s -type n -polynomial m -exponentially pre-invex function.

Proof. For every $\Theta, \mu_2 \in U, s \in [0, 1], w \in [0, 1], n \in \mathbb{N}, \theta \in \mathfrak{R}$, we have

$$\begin{aligned} & (\phi \circ \varphi)(m\mu_2 + w\xi(\Theta, \mu_2, m)) = \phi((\varphi)(m\mu_2 + w\xi(\Theta, \mu_2, m))) \\ & \leq \phi \left[\frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{\varphi(\Theta)}{e^{\theta\Theta}} + \frac{1}{n} \sum_{i=1}^n m^i (1 - (sw)^i) \frac{\varphi(\mu_2)}{e^{\theta\mu_2}} \right] \\ & \leq \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{\phi(\varphi(\Theta))}{e^{\theta\Theta}} + \frac{1}{n} \sum_{i=1}^n m^i (1 - (sw)^i) \frac{\phi(\varphi(\mu_2))}{e^{\theta\mu_2}} \\ & = \frac{1}{n} \sum_{i=1}^n (1 - (s(1-w))^i) \frac{(\phi \circ \varphi)(\Theta)}{e^{\theta\Theta}} + \frac{1}{n} \sum_{i=1}^n m^i (1 - (sw)^i) \frac{(\phi \circ \varphi)(\mu_2)}{e^{\theta\mu_2}}, \end{aligned}$$

□

which is required proof.

4. Behavior of H-H inequalities of s -type n -polynomial m -exponentially pre-invex function

Theorem 4.1. Let $U \subseteq \mathbb{R}$ be an open m -invex set with respect to the mapping $\xi : U \times U \times (0, 1] \rightarrow \mathbb{R}$, where $m, s \in (0, 1]$. Let $\Theta, \mu_2 \in U$ with $\Theta < \mu_2$, and assume that $m\mu_2 + \xi(\Theta, \mu_2, m) \leq m\mu_2$. Suppose that $\varphi : [m\mu_2 + \xi(\Theta, \mu_2, m), \mu_2] \rightarrow \mathbb{R}$ satisfies the extended Condition C. Then the following Hermite–Hadamard-type inequality holds:

$$\begin{aligned} \varphi \left(m\mu_2 + \frac{1}{2} \xi(\Theta, \mu_2, m) \right) & \leq \frac{1}{n \xi(\Theta, \mu_2, m)} \sum_{i=1}^n \left(1 - \left(\frac{s}{2} \right)^i \right) \\ & \quad \times \left[\int_{m\mu_2}^{m\mu_2 + \xi(\Theta, \mu_2, m)} \frac{\varphi(x)}{e^{\theta x}} dx + m^{i+1} \int_{\mu_2}^{\mu_2 + \frac{1}{m} \xi(\Theta, \mu_2, m)} \frac{\varphi(x)}{e^{\theta x}} dx \right] \\ & \leq \frac{1}{n} \sum_{i=1}^n \left(1 - \frac{s^i}{i+1} \right) \left[\frac{\varphi(\Theta)}{e^{\theta\Theta}} + m^i \frac{\varphi(\mu_2)}{e^{\theta\mu_2}} \right]. \end{aligned}$$

Proof. Since U is an m -invex set with respect to ξ , for every $m \in (0, 1]$ and $w \in (0, 1]$, we have

$$m\mu_2 + w \xi(\Theta, \mu_2, m) \in U.$$

Applying the extended Condition C for ξ , taking $w = \frac{1}{2}$, and utilizing Definition 3.1, we obtain the desired relation. Integrating both sides with respect to w over the interval $[0, 1]$ then yields the required inequality.

$$\begin{aligned} \varphi(my + w\xi(x, y, m)) & \leq \frac{1}{n} \sum_{i=1}^n (1 - (s(1-\theta))^i) \frac{\varphi(x)}{e^{\theta x}} + \frac{1}{n} \sum_{i=1}^n m^i (1 - (s\theta)^i) \frac{\varphi(y)}{e^{\theta y}} \\ \varphi(my + \frac{1}{2} \xi(x, y, m)) & \leq \frac{1}{n} \sum_{i=1}^n \left(1 - \left(\frac{s}{2} \right)^i \right) \frac{\varphi(x)}{e^{\theta x}} + \frac{1}{n} \sum_{i=1}^n m^i \left(1 - \left(\frac{s}{2} \right)^i \right) \frac{\varphi(y)}{e^{\theta y}} \\ \varphi(my + \frac{1}{2} \xi(x, y, m)) & \leq \frac{1}{n} \sum_{i=1}^n \left(1 - \left(\frac{s}{2} \right)^i \right) \left[\frac{\varphi(x)}{e^{\theta x}} + m^i \frac{\varphi(y)}{e^{\theta y}} \right] \end{aligned}$$

in above inequality we put $x = m\mu_2 + w\xi(\Theta, \mu_2, m)$ and $my = m\mu_2 + (1 - w)\xi(\Theta, \mu_2, m)$ so first we prove the Left Hand Side of the inequality

$$\varphi(my + \frac{1}{2}\xi(x, y, m)) = \varphi(m\mu_2 + \frac{1}{2}\xi(\Theta, \mu_2, m))$$

after putting the value of x and y , we obtain

$$\begin{aligned} \varphi(my + \frac{1}{2}\xi(x, y, m)) &= \varphi(my + \frac{1}{2}(x - my)) \\ &= \varphi(m\mu_2 + (1 - w)\xi(\Theta, \mu_2, m) + \frac{1}{2}(m\mu_2 + w\xi(\Theta, \mu_2, m) \\ &\quad - m\mu_2 - (1 - w)\xi(\Theta, \mu_2, m)) \end{aligned} \quad (21)$$

Now, by using the extended C condition, we have

$$\begin{aligned} \varphi(my + \frac{1}{2}\xi(x, y, m)) &= \varphi(m\mu_2 + (1 - w)\xi(\Theta, \mu_2, m) + \frac{1}{2}(w - 1 + w)\xi(\Theta, \mu_2, m)) \\ &= \varphi(m\mu_2 + (1 - w)\xi(\Theta, \mu_2, m) + \frac{1}{2}(2w - 1)\xi(\Theta, \mu_2, m)) \\ \varphi(my + \frac{1}{2}\xi(x, y, m)) &= \varphi(m\mu_2 + (1 - w)\xi(\Theta, \mu_2, m) + \frac{1}{2}(2w - 1)\xi(\Theta, \mu_2, m)) \\ \varphi(my + \frac{1}{2}\xi(x, y, m)) &= \varphi(m\mu_2 + (1 - w + w - \frac{1}{2})\xi(\Theta, \mu_2, m)) \\ \varphi(my + \frac{1}{2}\xi(x, y, m)) &= \varphi(m\mu_2 + \frac{1}{2}\xi(\Theta, \mu_2, m)) \end{aligned}$$

Now,

$$\begin{aligned} &\varphi(m\mu_2 + \frac{1}{2}\xi(\Theta, \mu_2, m)) \\ &\leq \frac{1}{n} \sum_{i=1}^n (1 - (\frac{s}{2})^i) \left[\int_0^1 \frac{\varphi(m\mu_2 + w\xi(\Theta, \mu_2, m))}{e^{\theta(m\mu_2 + w\xi(\Theta, \mu_2, m))}} dw + m^i \int_0^1 \frac{\varphi(\mu_2 + \frac{(1-w)}{m}\xi(\Theta, \mu_2, m))}{e^{\theta(\mu_2 + \frac{(1-w)}{m}\xi(\Theta, \mu_2, m))}} dw \right] \end{aligned}$$

By using substitutions $x = m\mu_2 + w\xi(\Theta, \mu_2, m) \Rightarrow dw = \frac{dx}{\xi(\Theta, \mu_2, m)}$, where $w = 0 \Rightarrow x = m\mu_2$ and $w = 1 \Rightarrow x = m\mu_2 + \xi(\Theta, \mu_2, m)$

as $x = \mu_2 + \frac{(1-w)}{m}\xi(\Theta, \mu_2, m) \Rightarrow dw = \frac{-mdx}{\xi(\Theta, \mu_2, m)}$, where $w = 0 \Rightarrow x = \mu_2 + \frac{1}{m}\xi(\Theta, \mu_2, m)$ and $w = 1 \Rightarrow x = \mu_2$, we have

$$\begin{aligned} \varphi(m\mu_2 + \frac{1}{2}\xi(\Theta, \mu_2, m)) &\leq \frac{1}{n} \frac{\sum_{i=1}^n (1 - (\frac{s}{2})^i)}{\xi(\Theta, \mu_2, m)} \left[\int_{m\mu_2}^{m\mu_2 + \xi(\Theta, \mu_2, m)} \frac{\varphi(x)}{e^{\theta x}} dx \right. \\ &\quad \left. + m^{i+1} \int_{\mu_2}^{\mu_2 + \frac{1}{m}\xi(\Theta, \mu_2, m)} \frac{\varphi(x)}{e^{\theta x}} dx \right] \end{aligned}$$

To establish the right-hand side of the inequality, we apply the definition of an s -type n -polynomial m -exponentially pre-invex function to obtain

$$\int_0^1 \varphi(m\mu_2 + w\xi(\Theta, \mu_2, m)) dw \leq \frac{\varphi(\Theta)}{ne^{\theta\Theta}} \sum_{i=1}^n \int_0^1 (1 - (s(1-w))^i) dw + \frac{\varphi(\mu_2)}{ne^{\theta\mu_2}} \sum_{i=1}^n m^i \int_0^1 (1 - (sw)^i) dw$$

Since,

$$\begin{aligned} \int_0^1 \varphi(m\mu_2 + w\xi(\Theta, \mu_2, m))dw &= \frac{1}{\xi(\Theta, \mu_2, m)} \int_{m\mu_2}^{m\mu_2 + \xi(\Theta, \mu_2, m)} \varphi(x)dx \\ \frac{1}{\xi(\Theta, \mu_2, m)} \int_{m\mu_2}^{m\mu_2 + \xi(\Theta, \mu_2, m)} \varphi(x)dx &\leq \frac{\varphi(\Theta)}{ne^{\theta\Theta}} \sum_{i=1}^n \int_0^1 (1 - (s(1-w))^i)dw \\ &+ \frac{\varphi(\mu_2)}{ne^{\theta\mu_2}} \sum_{i=1}^n m^i \int_0^1 (1 - (sw)^i)dw \quad (22) \\ \int_0^1 \varphi(\mu_2 + \frac{(1-w)}{m}\xi(\Theta, \mu_2, m))dw &\leq \frac{\varphi(\Theta)}{ne^{\theta\Theta}} \sum_{i=1}^n \int_0^1 (1 - (sw)^i)dw \\ &+ \frac{\varphi(\mu_2)}{ne^{\theta\mu_2}} \sum_{i=1}^n m^i \int_0^1 (1 - (s(1-w))^i)dw \end{aligned}$$

As,

$$\begin{aligned} \int_0^1 \varphi(\mu_2 + \frac{(1-w)}{m}\xi(\Theta, \mu_2, m))dw &= \frac{m^{i+1}}{\xi(\Theta, \mu_2, m)} \int_{\mu_2}^{\mu_2 + \frac{1}{m}\xi(\Theta, \mu_2, m)} \varphi(x)dx \\ \frac{m^{i+1}}{\xi(\Theta, \mu_2, m)} \int_{\mu_2}^{\mu_2 + \frac{1}{m}\xi(\Theta, \mu_2, m)} \varphi(x)dx &\leq \frac{\varphi(\Theta)}{ne^{\theta\Theta}} \sum_{i=1}^n \int_0^1 (1 - (sw)^i)dw \\ &+ \frac{\varphi(\mu_2)}{ne^{\theta\mu_2}} \sum_{i=1}^n m^i \int_0^1 (1 - (s(1-w))^i)dw \quad (23) \end{aligned}$$

Adding equation (22) and equation (23), we have

$$\begin{aligned} &\frac{1}{\xi(\Theta, \mu_2, m)} \left[\int_{m\mu_2}^{m\mu_2 + \xi(\Theta, \mu_2, m)} \varphi(x)dx + m^{i+1} \int_{\mu_2}^{\mu_2 + \frac{1}{m}\xi(\Theta, \mu_2, m)} \varphi(x)dx \right] \\ &= \int_0^1 \varphi(m\mu_2 + w\xi(\Theta, \mu_2, m))dw + \int_0^1 \varphi(\mu_2 + \frac{(1-w)}{m}\xi(\Theta, \mu_2, m))dw \\ &\leq \frac{\varphi(\Theta)}{ne^{\theta\Theta}} \sum_{i=1}^n \int_0^1 (1 - (s(1-w))^i)dw + \frac{\varphi(\mu_2)}{ne^{\theta\mu_2}} \sum_{i=1}^n m^i \int_0^1 (1 - (sw)^i)dw \\ &+ \frac{\varphi(\Theta)}{ne^{\theta\Theta}} \sum_{i=1}^n \int_0^1 (1 - (sw)^i)dw + \frac{\varphi(\mu_2)}{ne^{\theta\mu_2}} \sum_{i=1}^n m^i \int_0^1 (1 - (s(1-w))^i)dw \\ &\leq \frac{2}{n} \sum_{i=1}^n \left(1 - \frac{s^i}{i+1}\right) \left[\frac{\varphi(\Theta)}{e^{\theta\Theta}} + m^i \frac{\varphi(\mu_2)}{e^{\theta\mu_2}} \right]. \end{aligned}$$

This complete the proof. $\square$

Corollary 4.2. By setting $m = 1$ and $\theta = 0$ in Theorem 4.1, we recover the Hermite–Hadamard inequality for s -type n -polynomial pre-invex functions, as presented in [21].

Corollary 4.3. By taking $n = 1$, $s = 1$, $m = 1$, $\theta = 0$, and $\xi(\Theta, \mu_2, m) = \Theta - \mu_2$ in Theorem 4.1, we recover the classical Hermite–Hadamard inequality.

Corollary 4.4. By setting $s = 1$, $m = 1$, $\theta = 0$, and $\xi(\Theta, \mu_2, m) = \Theta - \mu_2$ in Theorem 4.1, we recover inequality (3.1) established by Toplu et al. [11].

5. Conclusions

In this research work, we introduced the concept of an s -type n -polynomial m -exponentially pre-invex function and established several fundamental properties associated with this class of functions under a non-decreasing function ξ . We further developed the (s, n, m) -pre-invex framework and investigated its algebraic properties and potential applications. Based on these generalized convexity concepts, we derived several new Hermite–Hadamard-type inequalities and established meaningful refinements of existing inequalities. Furthermore, by selecting appropriate values of the involved parameters, our results yield several well-known inequalities and corresponding corollaries available in the recent literature. These special cases demonstrate the generality of the proposed framework and highlight the significance of the obtained results.

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