



Research Article

A Novel Double-Application HA-Iteration for Generalized (α, β) -Nonexpansive Mappings: Fixed Point Structure and Convergence Theory

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Abstract

We study generalized (α, β) -nonexpansive $(G(\alpha, \beta)NE)$ mappings Q on Banach space subsets, approximated via the HA-iteration, a three-step scheme using Q^2 in its auxiliary steps and Q only in the final step. As $G(\alpha, \beta)NE$ is Suzuki-type, Q^2 need not inherit membership, and $F(Q)$, $F(Q^2)$ need not coincide; we prove $F(Q) \subseteq F(Q^2)$ always, with possible strict inclusion, and isolate the hypotheses — square-regularity (H) and fixed-set coincidence (E) — required for the HA-iteration theory (asymptotic-center estimates, boundedness/existence for $F(Q^2)$, demiclosedness of $I - Q^2$, Fejér monotonicity, strong and weak convergence) to hold rigorously. A numerical example satisfying (H) and (E) illustrates the scheme's behavior against classical iterations.

1. Introduction

Fixed point theory sits at the meeting point of geometry, topology, and analysis; its methods routinely settle existence and uniqueness questions for integral and differential equations and variational inequalities. Browder [1], Kirk [2], and Göhde [3] showed independently in 1965 that a nonexpansive self-map of a closed, bounded, convex subset of a uniformly convex Banach space has a fixed point, triggering a substantial literature on fixed point approximation in uniformly convex Banach spaces; see [4, 5].

Recall that $Q : D \rightarrow D$ is a *contraction* if $\|Q\mu - Q\nu\| \leq \alpha\|\mu - \nu\|$ for some $\alpha \in [0, 1)$ and all $\mu, \nu \in D$; the Banach Contraction Principle produces its unique fixed point as the limit of the Picard iteration $\mu_{n+1} = Q\mu_n$. When Q is only nonexpansive, i.e. $\|Q\mu - Q\nu\| \leq \|\mu - \nu\|$, Picard iterates need not converge, and existence of a fixed point instead rests on uniform convexity.

For $\mu, \nu \in D$, Q is called: (i) *Suzuki nonexpansive* if $\frac{1}{2}\|\mu - Q\nu\| \leq \|\mu - \nu\| \Rightarrow \|Q\mu - Q\nu\| \leq \|\mu - \nu\|$; (ii) α -*nonexpansive* if $\|Q\mu - Q\nu\|^2 \leq \alpha\|\mu - Q\nu\|^2 + \alpha\|\nu - Q\mu\|^2 + (1 - 2\alpha)\|\mu - \nu\|^2$; (iii) *generalized α -nonexpansive* if $\frac{1}{2}\|\mu - Q\mu\| \leq \|\mu - \nu\| \Rightarrow \|Q\mu - Q\nu\| \leq \alpha\|\nu - Q\mu\| + \alpha\|\mu - Q\nu\| + (1 - 2\alpha)\|\mu - \nu\|$; (iv) *Reich–Suzuki nonexpansive* if $\frac{1}{2}\|\mu - Q\mu\| \leq \|\mu - \nu\| \Rightarrow \|Q\mu - Q\nu\| \leq \beta\|\mu - Q\mu\| + \beta\|\nu - Q\nu\| + (1 - 2\beta)\|\mu - \nu\|$.

Remark 1.1. Every nonexpansive mapping is Suzuki nonexpansive, and every Suzuki nonexpansive mapping is generalized α -nonexpansive and Reich–Suzuki nonexpansive (with $\alpha = \beta = 0$).

Ullah et al. [6] unified these classes in 2020 by introducing the $G(\alpha, \beta)$ NE mappings (Definition 2.1 below) and approximated their fixed points through a K -iteration; like the Mann, Ishikawa, Noor, Agarwal, Abbas–Nazir, Thakur, M , and HK-type schemes [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] before and after it, each of these builds every auxiliary point from at most *one* application of Q . We instead study the **HA-iteration** of [18], in which both auxiliary points are built from a *double* application $Q^2 := Q \circ Q$:

$$\begin{cases} g_h = (1 - \mu_h)t_h + \mu_h Q^2 t_h, \\ j_h = (1 - \lambda_h)Q^2 t_h + \lambda_h Q^2 g_h, & \{\lambda_h\}_{h \geq 0}, \{\mu_h\}_{h \geq 0} \subset [0, 1]. \\ t_{h+1} = Qj_h, \end{cases} \quad (1.1)$$

Here α, β are reserved exclusively for the constants in the $G(\alpha, \beta)$ NE condition, and λ_h, μ_h exclusively for the control sequences of (1.1); the two pairs of symbols are never interchanged.

The role of Q^2 . Since g_h, j_h are formed from Q^2 and only t_{h+1} applies Q once, the set naturally tied to (1.1) is $F(Q^2) = \{x \in D : Q^2 x = x\}$, not $F(Q) = \{x \in D : Qx = x\}$. Because the defining inequality of $G(\alpha, \beta)$ NE is a *conditional* (Suzuki-type) inequality rather than a genuine Lipschitz bound, membership of Q in $G(\alpha, \beta)$ NE does *not* automatically imply that Q^2 is again $G(\alpha, \beta)$ NE, and $F(Q) = F(Q^2)$ need not hold. Both facts are false in general even for the classical nonexpansive subclass (Examples 2.3 and 2.4 below). The present paper's central mathematical content is to isolate the exact extra hypotheses under which the double-application machinery is rigorous, rather than assuming inheritance silently.

Contribution. We (1) give precise definitions of $F(Q)$, $F(Q^2)$ and prove the correct relationship between them (Remark 2.2, Examples 2.3–2.4); (2) isolate Assumption (H) (square-regularity: $Q^2 \in G(\alpha, \beta)$ NE) and Assumption (E) ($F(Q) = F(Q^2)$), give a sufficient condition for each, and use them explicitly and only where logically required; (3) prove, under (H), an asymptotic-center estimate for Q^2 (Lemma 2.12) and a demiclosed principle for $I - Q^2$ (Corollary 2.14); (4) prove a genuine existence criterion for $F(Q^2)$ (Theorem 3.3); (5) prove strong convergence (via compactness and via Condition I) and weak convergence theorems for (1.1)

under (H) and (E), with every use of a fixed point attributed correctly to $F(Q)$ or $F(Q^2)$; and (6) give a reproducible numerical example, verifying (H) and (E) explicitly, comparing (1.1) against Picard, Mann, and a K -type iteration.

Novelty relative to HA/HK-type schemes. Table 1 compares (1.1) with the single-application HK-type schemes of [6, 13, 14, 15, 16, 17] and with the enriched-mapping HA-iteration of [18]. Unlike [18], which iterates an enriched nonexpansive mapping and needs no distinction between Q and Q^2 , the present scheme applies a genuinely different operator (Q^2 inside, Q at closure) to the strictly larger $G(\alpha, \beta)$ NE class, which is exactly what forces the $F(Q)$ vs. $F(Q^2)$ analysis above; this analysis, and Assumptions (H)/(E) that make it rigorous, are the new elements not present in [6, 13, 14, 15, 16, 17, 18]. When Q is nonexpansive, both (H) and (E) hold automatically (Propositions 2.6 and the discussion after Remark 2.2) and every theorem below recovers a corresponding classical statement for the single-application schemes as a special case. We make no unquantified claim that (1.1) is “faster” than these schemes; Section 4 instead reports a concrete, reproducible comparison.

Table 1: Structural comparison of iteration schemes (D closed convex, $\lambda_h, \mu_h, \gamma_h \in (0, 1)$).

Scheme	Applications of Q per step	Auxiliary points from	Natural limit set
Mann [7]	1	t_h	$F(Q)$
K -iteration [6]	2 (single each)	Qt_h, Qg_h	$F(Q)$
HK-iteration [13, 17]	2 (single each)	Qt_h, Qg_h	$F(Q)$
Enriched HA [18]	3 (single each, enriched map)	t_h, g_h	$F(Q)$
Present HA (1.1)	3, with Q^2 twice	Q^2t_h, Q^2g_h	$F(Q^2) \supseteq F(Q)$

2. Preliminaries

A Banach space B is *uniformly convex* (UC) [19] if for every $\varepsilon \in [0, 1)$ there is $\delta > 0$ such that $\|\mu\| \leq 1, \|\nu\| \leq 1, \|\mu - \nu\| \geq \varepsilon$ imply $\|\mu + \nu\|/2 \leq 1 - \delta$. It satisfies the *Opial condition* [20] if $\mu_i \rightharpoonup w$ implies $\liminf_i \|\mu_i - w\| < \liminf_i \|\mu_i - w'\|$ for every $w' \neq w$. For $\emptyset \neq D \subset B$ and $\{\mu_i\}$ bounded in B , the asymptotic radius at $\mu \in D$ is $A_r(\mu, \{\mu_i\}) = \limsup_i \|\mu - \mu_i\|$, the radius relative to D is $A_r(D, \{\mu_i\}) = \inf_{\mu \in D} A_r(\mu, \{\mu_i\})$, and the asymptotic center $A_c(D, \{\mu_i\})$ is the set of minimizers. When B is UC, $A_c(D, \{\mu_i\})$ is a single point, and it is nonempty and convex whenever D is weakly compact and convex [21, 22].

Throughout, D is a nonempty closed and convex (c.c.) subset of a Banach space B , $Q : D \rightarrow D$, and

$$F(Q) = \{x \in D : Qx = x\}, \quad F(Q^2) = \{x \in D : Q^2x = x\}, \quad Q^2 := Q \circ Q.$$

Definition 2.1. Q is $G(\alpha, \beta)$ NE if there are $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that, for all $\mu, \nu \in D$,

$$\begin{aligned} \frac{1}{2}\|\mu - Q\mu\| &\leq \|\mu - \nu\| \implies \|Q\mu - Q\nu\| \\ &\leq \alpha\|\mu - Q\nu\| + \alpha\|\nu - Q\mu\| + \beta\|\mu - Q\mu\| + \beta\|\nu - Q\nu\| + (1 - 2\alpha - 2\beta)\|\mu - \nu\|. \end{aligned}$$

This class properly contains the Suzuki, generalized α -, and Reich–Suzuki nonexpansive mappings [6].

Remark 2.2 ($F(Q) \subseteq F(Q^2)$, and no more in general). If $p \in F(Q)$ then $Q^2p = Q(Qp) = Qp = p$, so $F(Q) \subseteq F(Q^2)$. If moreover Q is $G(\alpha, \beta)$ NE, Lemma 2.9 below makes Q quasi-nonexpansive at p , i.e. $\|Qx - p\| \leq \|x - p\|$ for all $x \in D$, whence $\|Q^2x - p\| \leq \|Qx - p\| \leq$

$\|x - p\|$: $\mathcal{Q}^2$ is quasi-nonexpansive at p too. This is the only inheritance available without further hypotheses.

Example 2.3. Let $D = [-1, 1] \subset \mathbb{R}$ and $\mathcal{Q}(x) = -x$. Then $\mathcal{Q}$ is an isometry, hence nonexpansive and $G(0, 0)NE$. Here $F(\mathcal{Q}) = \{0\}$ while $\mathcal{Q}^2 = \mathbb{1}$, so $F(\mathcal{Q}^2) = D = [-1, 1]$. Thus $F(\mathcal{Q}) \subsetneq F(\mathcal{Q}^2)$ even when $\mathcal{Q}$ is nonexpansive: inclusion in Remark 2.2 is generally strict.

Example 2.4. Let $D = [0, 1]$ and $\mathcal{Q}(x) = 0$ for $x \neq 1$, $\mathcal{Q}(1) = 1/2$. This is Suzuki's classical example [23] of a $G(0, 0)NE$ mapping that is discontinuous, hence not nonexpansive. Direct computation gives $\mathcal{Q}^2 \equiv 0$ on D , a constant (hence trivially nonexpansive, so $G(0, 0)NE$) map; here $F(\mathcal{Q}) = F(\mathcal{Q}^2) = \{0\}$. This shows $G(\alpha, \beta)NE$ -ness of $\mathcal{Q}^2$ can hold even when $\mathcal{Q}$ itself is not continuous, so inheritance must be checked case by case rather than assumed either way.

Assumption 2.5 (Square-regularity). (H): $\mathcal{Q}^2$ is itself a $G(\alpha, \beta)NE$ self-map of D (for the same constants α, β , without loss of generality).

Proposition 2.6. If $\mathcal{Q}$ is nonexpansive then $\mathcal{Q}^2$ is nonexpansive, hence $G(0, 0)NE$; thus (H) holds automatically whenever $\mathcal{Q}$ is nonexpansive. In general, (H) is not implied by $\mathcal{Q} \in G(\alpha, \beta)NE$: the defining implication for $\mathcal{Q}^2$ at μ, ν would require the hypothesis $\frac{1}{2}\|\mathcal{Q}\mu - \mathcal{Q}^2\mu\| \leq \|\mathcal{Q}\mu - \mathcal{Q}\nu\|$, which is not guaranteed merely because $\frac{1}{2}\|\mu - \mathcal{Q}^2\mu\| \leq \|\mu - \nu\|$ holds; so (H) is a genuine additional hypothesis, to be verified for the mapping at hand (Example 2.4 verifies it directly for a discontinuous mapping).

Proof. $\|\mathcal{Q}^2\mu - \mathcal{Q}^2\nu\| \leq \|\mathcal{Q}\mu - \mathcal{Q}\nu\| \leq \|\mu - \nu\|$ by two applications of nonexpansiveness. $\square$

Assumption 2.7 (Fixed point set coincidence). (E): $F(\mathcal{Q}) = F(\mathcal{Q}^2)$.

Proposition 2.8. Assumption (E) holds if and only if $\mathcal{Q}$ is quasi-nonexpansive relative to every point of $F(\mathcal{Q}^2)$, i.e. $\|\mathcal{Q}x - p\| \leq \|x - p\|$ for all $x \in D$ and $p \in F(\mathcal{Q}^2)$. In particular (E) holds whenever $F(\mathcal{Q}^2)$ has this property, and Example 2.3 shows (E) can fail even for a nonexpansive, hence quasi-nonexpansive-at-its-own-fixed-points, isometry.

Proof. If the stated quasi-nonexpansivity holds, take $x = p \in F(\mathcal{Q}^2)$: $\|\mathcal{Q}p - p\| \leq \|p - p\| = 0$, so $\mathcal{Q}p = p$, i.e. $p \in F(\mathcal{Q})$; with Remark 2.2 this gives $F(\mathcal{Q}^2) \subseteq F(\mathcal{Q}) \subseteq F(\mathcal{Q}^2)$, i.e. (E). Conversely if (E) holds, every $p \in F(\mathcal{Q}^2)$ lies in $F(\mathcal{Q})$ and Lemma 2.9 gives the stated inequality. $\square$

The following two lemmas, established in [6], are used repeatedly.

Lemma 2.9. If $\mathcal{Q}$ is $G(\alpha, \beta)NE$ with $F(\mathcal{Q}) \neq \emptyset$, then $\mathcal{Q}$ is quasi-nonexpansive: $\|\mathcal{Q}x - p\| \leq \|x - p\|$ for all $x \in D$, $p \in F(\mathcal{Q})$.

Lemma 2.10. If $\mathcal{Q}$ is $G(\alpha, \beta)NE$ then $F(\mathcal{Q})$ is closed; if in addition D is convex and B strictly convex, $F(\mathcal{Q})$ is convex.

Lemma 2.11. If $\mathcal{Q}$ is $G(\alpha, \beta)NE$, then for all $\mu, \nu \in D$, $\|\mu - \mathcal{Q}\nu\| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)\|\mu - \mathcal{Q}\mu\| + \|\mu - \nu\|$.

Lemma 2.12. Under (H), for all $\mu, \nu \in D$, $\|\mu - \mathcal{Q}^2\nu\| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)\|\mu - \mathcal{Q}^2\mu\| + \|\mu - \nu\|$.

Proof. By (H), $\mathcal{Q}^2$ is $G(\alpha, \beta)NE$, and the statement is exactly Lemma 2.11 applied to the mapping $\mathcal{Q}^2$ in place of $\mathcal{Q}$. $\square$

Lemma 2.13. *Let D have the Opial property and $\mathcal{Q}$ be $G(\alpha, \beta)$ NE. If $\{\mu_i\} \subset D$, $\mu_i \rightharpoonup w$ and $\|\mu_i - \mathcal{Q}\mu_i\| \rightarrow 0$, then $\mathcal{Q}w = w$ ([6]).*

Corollary 2.14. *Under (H), if $\{\mu_i\} \subset D$, $\mu_i \rightharpoonup w$, and $\|\mu_i - \mathcal{Q}^2\mu_i\| \rightarrow 0$, then $\mathcal{Q}^2w = w$, i.e. $I - \mathcal{Q}^2$ is demiclosed at 0.*

Proof. By (H), $\mathcal{Q}^2$ is $G(\alpha, \beta)$ NE and D retains the Opial property (a property of the space, not of the map); apply Lemma 2.13 to $\mathcal{Q}^2$. $\square$

Lemma 2.15 ([24]). *Let B be UC and $0 < s \leq k_i \leq t < 1$ for all i . If $\{\nu_i\}, \{z_i\} \subset B$ satisfy $\limsup \|\nu_i\| \leq \gamma$, $\limsup \|z_i\| \leq \gamma$, and $\lim \|k_i\nu_i + (1 - k_i)z_i\| = \gamma$ for some $\gamma \geq 0$, then $\lim \|\nu_i - z_i\| = 0$.*

3. Convergence Theorems in Uniformly Convex Banach Spaces

Throughout, B is UC, $D \subset B$ is c.c., $\mathcal{Q} : D \rightarrow D$ is $G(\alpha, \beta)$ NE, and $\{t_h\}$ is generated by (1.1).

Lemma 3.1. *If $F(\mathcal{Q}) \neq \emptyset$, then for every $p \in F(\mathcal{Q})$, $\lim_{h \rightarrow \infty} \|t_h - p\|$ exists.*

Proof. Fix $p \in F(\mathcal{Q})$. By Lemma 2.9 and Remark 2.2, $\|\mathcal{Q}^2x - p\| \leq \|x - p\|$ for all $x \in D$. Then $\|g_h - p\| \leq (1 - \mu_h)\|t_h - p\| + \mu_h\|\mathcal{Q}^2t_h - p\| \leq \|t_h - p\|$, $\|j_h - p\| \leq (1 - \lambda_h)\|\mathcal{Q}^2t_h - p\| + \lambda_h\|\mathcal{Q}^2g_h - p\| \leq (1 - \lambda_h)\|t_h - p\| + \lambda_h\|g_h - p\| \leq \|t_h - p\|$, and, since $p = \mathcal{Q}p \in F(\mathcal{Q})$, $\|t_{h+1} - p\| = \|\mathcal{Q}j_h - p\| \leq \|j_h - p\| \leq \|t_h - p\|$. So $\{\|t_h - p\|\}$ is nonincreasing and bounded below, hence convergent. $\square$

Remark 3.2. *The last step, $\|\mathcal{Q}j_h - p\| \leq \|j_h - p\|$, uses $p \in F(\mathcal{Q})$, not merely $p \in F(\mathcal{Q}^2)$: if $p \in F(\mathcal{Q}^2) \setminus F(\mathcal{Q})$ then $q := \mathcal{Q}p \neq p$ satisfies $\mathcal{Q}q = p$, and quasi-nonexpansivity of $\mathcal{Q}$ at p is not implied by $\mathcal{Q} \in G(\alpha, \beta)$ NE; by Proposition 2.8 it holds for every $p \in F(\mathcal{Q}^2)$ exactly when (E) holds. This is why Lemma 3.1 is stated, and only stated, for $p \in F(\mathcal{Q})$; every later use of it for $p \in F(\mathcal{Q}^2)$ is licensed by (E), under which $F(\mathcal{Q}^2) = F(\mathcal{Q})$.*

Theorem 3.3. *Let $F(\mathcal{Q}) \neq \emptyset$ and $\{\lambda_h\}, \{\mu_h\} \subset [a, b] \subset (0, 1)$. Then $\{t_h\}$ is bounded and $\lim_{h \rightarrow \infty} \|t_h - \mathcal{Q}^2t_h\| = 0$. Conversely, under (H), if $\{t_h\}$ is a bounded sequence generated by (1.1) with $\lim_{h \rightarrow \infty} \|t_h - \mathcal{Q}^2t_h\| = 0$, then $F(\mathcal{Q}^2) \neq \emptyset$.*

Proof. Let $p \in F(\mathcal{Q})$; by Lemma 3.1, $\eta := \lim_h \|t_h - p\|$ exists and $\{t_h\}$ is bounded. From the proof of Lemma 3.1, $\limsup \|g_h - p\| \leq \eta$ and $\limsup \|\mathcal{Q}^2t_h - p\| \leq \eta$. Since $\|t_{h+1} - p\| \leq \|j_h - p\| \leq \|t_h - p\|$ and both outer terms tend to η , $\lim \|j_h - p\| = \eta$; comparing with $\|j_h - p\| \leq (1 - \lambda_h)\|t_h - p\| + \lambda_h\|g_h - p\|$ (whose $\limsup \leq \eta$) forces $\lim[(1 - \lambda_h)\|t_h - p\| + \lambda_h\|g_h - p\|] = \eta$, and since $\|t_h - p\| \rightarrow \eta$, $\lambda_h \geq a > 0$, we get $\lim \|g_h - p\| = \eta$. Applying Lemma 2.15 to $g_h = (1 - \mu_h)t_h + \mu_h\mathcal{Q}^2t_h$ with $\mu_h \in [a, b]$ gives $\lim \|t_h - \mathcal{Q}^2t_h\| = 0$.

Conversely, assume (H), boundedness, and $\|t_h - \mathcal{Q}^2t_h\| \rightarrow 0$. Let $p \in A_c(D, \{t_h\})$; by Lemma 2.12,

$$\begin{aligned} A_r(\mathcal{Q}^2p, \{t_h\}) &= \limsup_h \|t_h - \mathcal{Q}^2p\| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right) \limsup_h \|\mathcal{Q}^2t_h - t_h\| + \limsup_h \|t_h - p\| \\ &= A_r(p, \{t_h\}). \end{aligned}$$

So $\mathcal{Q}^2p \in A_c(D, \{t_h\})$, a singleton since B is UC; hence $\mathcal{Q}^2p = p$, i.e. $F(\mathcal{Q}^2) \neq \emptyset$. $\square$

Remark 3.4. *By Remark 2.2, $F(\mathcal{Q}) \subseteq F(\mathcal{Q}^2)$ always; Theorem 3.3 shows boundedness of $\{t_h\}$ together with $\|t_h - \mathcal{Q}^2t_h\| \rightarrow 0$ characterizes $F(\mathcal{Q}^2) \neq \emptyset$ under (H), not $F(\mathcal{Q}) \neq \emptyset$. It becomes a genuine if-and-only-if statement about $F(\mathcal{Q})$ precisely when (E) also holds.*

Theorem 3.5. Let D be compact, $F(\mathcal{Q}) \neq \emptyset$, $\{\lambda_h\}, \{\mu_h\} \subset [a, b] \subset (0, 1)$, and assume (H) and (E). Then $\{t_h\}$ converges strongly to a point of $F(\mathcal{Q}^2) = F(\mathcal{Q})$.

Proof. By Theorem 3.3, $\|\mathcal{Q}^2 t_h - t_h\| \rightarrow 0$. By compactness, some subsequence $t_{h_k} \rightarrow s \in D$. By Lemma 2.12, $\|t_{h_k} - \mathcal{Q}^2 s\| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right) \|t_{h_k} - \mathcal{Q}^2 t_{h_k}\| + \|t_{h_k} - s\| \rightarrow 0$, and since strong limits are unique, $\mathcal{Q}^2 s = s$, so $s \in F(\mathcal{Q}^2)$. By (E), $s \in F(\mathcal{Q})$, so Lemma 3.1 applies at $p = s$: $\lim_h \|t_h - s\|$ exists, and since a subsequence tends to 0, the whole sequence does; $t_h \rightarrow s$. $\square$

Theorem 3.6. Let $F(\mathcal{Q}) \neq \emptyset$, assume (E), $\{\lambda_h\}, \{\mu_h\} \subset [a, b] \subset (0, 1)$, and $\liminf_{h \rightarrow \infty} \text{dist}(t_h, F(\mathcal{Q}^2)) = 0$. Then $t_h \rightarrow t^*$ strongly for some $t^* \in F(\mathcal{Q}^2) = F(\mathcal{Q})$.

Proof. Write $F := F(\mathcal{Q}) = F(\mathcal{Q}^2) \neq \emptyset$ (by (E)), closed by Lemma 2.10. For each $p \in F$, Lemma 3.1 gives $\|t_{h+1} - p\| \leq \|t_h - p\|$; taking the infimum over $p \in F$, $\text{dist}(t_{h+1}, F) \leq \text{dist}(t_h, F)$, so $\{\text{dist}(t_h, F)\}$ is nonincreasing, and by hypothesis its $\liminf$ (hence its limit) is 0: $\text{dist}(t_h, F) \rightarrow 0$. Given $\varepsilon > 0$ pick N with $\text{dist}(t_N, F) < \varepsilon/4$, and $p \in F$ with $\|t_N - p\| < \varepsilon/4$. For $m, n \geq N$, $\|t_n - t_m\| \leq \|t_n - p\| + \|t_m - p\| \leq 2\|t_N - p\| < \varepsilon/2 < \varepsilon$ (using $\|t_h - p\|$ nonincreasing, Lemma 3.1). So $\{t_h\}$ is Cauchy, hence $t_h \rightarrow t^* \in D$; since $\text{dist}(t_h, F) \rightarrow 0$ and F is closed, $t^* \in F$. $\square$

Definition 3.7 ([25]). $\mathcal{Q}^2 : D \rightarrow D$ satisfies Condition I if there is nondecreasing $h_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $h_0(0) = 0$, $h_0(u) > 0$ for $u > 0$, such that $\|x - \mathcal{Q}^2 x\| \geq h_0(\text{dist}(x, F(\mathcal{Q}^2)))$ for all $x \in D$.

Theorem 3.8. Let $F(\mathcal{Q}) \neq \emptyset$, assume (E), $\{\lambda_h\}, \{\mu_h\} \subset [a, b] \subset (0, 1)$, and suppose $\mathcal{Q}^2$ satisfies Condition I. Then $t_h \rightarrow t^* \in F(\mathcal{Q}) = F(\mathcal{Q}^2)$.

Proof. By Theorem 3.3, $\|t_h - \mathcal{Q}^2 t_h\| \rightarrow 0$; Condition I gives $h_0(\text{dist}(t_h, F(\mathcal{Q}^2))) \leq \|t_h - \mathcal{Q}^2 t_h\| \rightarrow 0$, and monotonicity of h_0 with $h_0(0) = 0$, $h_0 > 0$ on $(0, \infty)$ forces $\liminf_h \text{dist}(t_h, F(\mathcal{Q}^2)) = 0$. Conclude by Theorem 3.6. $\square$

Theorem 3.9. Let D have the Opial property, $F(\mathcal{Q}) \neq \emptyset$, assume (H) and (E), and $\{\lambda_h\}, \{\mu_h\} \subset [a, b] \subset (0, 1)$. Then $t_h \rightarrow t^*$ for some $t^* \in F(\mathcal{Q}) = F(\mathcal{Q}^2)$.

Proof. Fix $p_0 \in F(\mathcal{Q})$; by Lemma 3.1, $\{t_h\}$ is bounded, and by Theorem 3.3, $\|t_h - \mathcal{Q}^2 t_h\| \rightarrow 0$. B UC $\Rightarrow$ reflexive, so some subsequence $t_{h_j} \rightharpoonup v_1 \in D$; by Corollary 2.14, $v_1 \in F(\mathcal{Q}^2) = F(\mathcal{Q})$ by (E). Suppose another subsequence $t_{h_k} \rightharpoonup v_2 \neq v_1$; likewise $v_2 \in F(\mathcal{Q})$. By (E), Lemma 3.1 applies at both $v_1, v_2 \in F(\mathcal{Q})$, so $\lim_h \|t_h - v_1\|$ and $\lim_h \|t_h - v_2\|$ both exist. By the Opial property applied to $t_{h_j} \rightharpoonup v_1$ and $t_{h_k} \rightharpoonup v_2$,

$$\begin{aligned} \lim_h \|t_h - v_1\| &= \lim_j \|t_{h_j} - v_1\| < \lim_j \|t_{h_j} - v_2\| = \lim_h \|t_h - v_2\| = \lim_k \|t_{h_k} - v_2\| \\ &< \lim_k \|t_{h_k} - v_1\| = \lim_h \|t_h - v_1\|, \end{aligned}$$

a contradiction. Hence every weakly convergent subsequence of $\{t_h\}$ has the same limit $t^* \in F(\mathcal{Q})$, and since B is reflexive and $\{t_h\}$ bounded, $t_h \rightarrow t^*$. $\square$

4. Numerical Experiment

Let $D = [0, 1]$ and $\mathcal{Q}(x) = x - x^2$. Since $\mathcal{Q}'(x) = 1 - 2x \in [-1, 1]$ on D , $\mathcal{Q}$ is nonexpansive (hence G(0, 0)NE), so (H) holds by Proposition 2.6. Direct computation shows $F(\mathcal{Q}) = \{0\}$, and numerically solving $\mathcal{Q}^2(x) = x$ on a fine grid of D yields the single root $x = 0$, so $F(\mathcal{Q}^2) = \{0\} = F(\mathcal{Q})$: (E) holds as well. Thus every hypothesis of Theorems 3.5–3.9 is met,

and Q is not a contraction (its derivative equals 1 at $x = 0$), so this is a genuine nonexpansive, non-contractive test case.

We compared Picard ($t_{h+1} = Qt_h$), Mann ($t_{h+1} = 0.5t_h + 0.5Qt_h$), a K -type two-step scheme with parameters 0.5, and the HA-iteration (1.1) with $\lambda_h = \mu_h = 0.5$, from three starting points, stopping when $|t_h| < 10^{-3}$. All code and iteration formulas match exactly those stated above; Table 2 reports the iteration count needed.

Table 2: Iterations to reach $|t_h| < 10^{-3}$ for $Q(x) = x - x^2$ on $[0, 1]$.

t_0	Picard	Mann ($a = 0.5$)	K -iteration	HA-iteration (1.1)
0.9	986	1991	793	284
0.5	992	1990	794	284
0.1	986	1976	789	282

On this example the double application of Q inside (1.1) empirically reduces the number of iterations needed by a factor of roughly 3 compared with the K -iteration and more than 6 compared with Picard iteration, consistent with (but not a substitute for) the qualitative convergence guaranteed by Theorems 3.5–3.9. We report only this measured behavior and make no general asymptotic-rate claim beyond it.

5. Conclusions

We gave precise definitions of $F(Q)$ and $F(Q^2)$ for the HA-iteration (1.1) on generalized (α, β) -nonexpansive mappings, proved $F(Q) \subseteq F(Q^2)$ with strict inclusion possible even for isometries (Example 2.3), and identified the exact extra hypotheses — square-regularity (H) and fixed-point-set coincidence (E) — needed to extend the asymptotic-center estimate, demiclosed principle, Fejér monotonicity, and strong/weak convergence theory from Q to Q^2 . Every theorem above states exactly which of (H), (E) it uses and where a fixed point is drawn from $F(Q)$ versus $F(Q^2)$. Both hypotheses hold automatically when Q is nonexpansive, recovering the classical theory for single-application schemes as a special case, and a reproducible numerical example confirms both hypotheses directly and illustrates the practical behavior of (1.1) against Picard, Mann, and K -type iterations. Since $G(\alpha, \beta)$ NE strictly contains the nonexpansive, Suzuki, Reich–Suzuki, and generalized α -nonexpansive classes, these results extend the corresponding fixed point approximation results in [1, 2, 3, 5, 12, 26, 27, 28, 29, 30].

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