



Research Article

Some New Regularity Properties of Elliptic Equations in Generalized Zygmund Spaces with VMO Coefficients

Waqar Afzal

Abdus Salam School of Mathematical Sciences, GC University, Lahore 54600, Pakistan

* Corresponding author: waqar_afzal_22@sms.edu.pk

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Abstract

Let $\mathcal{L}u = -\sum_{i,j} (a_{ij}u_{x_i})_{x_j} = \nabla \cdot \mathbf{h}$ on $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 3$, $(a_{ij}) \in \mathcal{V} \cap L^\infty$. For $1 < q < n$, $0 < \lambda < n$, $\beta \geq 0$, set $\Phi(t) = t^q \log^\beta(e+t)$ and let $\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})$ be the Φ -modulated ball-oscillation (Zygmund) class. For $\mathbf{h} \in [\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})]^n$ we prove $\partial_{x_i} u \in \mathcal{Z}_\beta^{q,\lambda}(K)$, $K \Subset \mathcal{D}$, with a quantitative bound, via a parametrix decomposition, a Calderón–Zygmund operator estimate on $\mathcal{Z}_\beta^{q,\lambda}$ obtained from the Orlicz-space singular-integral theory together with the norm comparison (6), an $\mathcal{V}$ -smallness estimate, and a Banach contraction. A Hölder corollary follows at $\lambda + q > n$.

1. Introduction

Calderón and Zygmund [1] founded the singular-integral theory underlying all subsequent regularity results for

$$\mathcal{L}u := - \sum_{i,j=1}^n (\mathfrak{a}_{ij}(x)u_{x_i})_{x_j} = \nabla \cdot \mathbf{h}(x), \quad x \in \mathcal{D} \subset \mathbb{R}^n, \quad n \geq 3. \quad (1)$$

Chiarenza, Frasca and Longo [2] proved $W^{2,q}$ -solvability of the non-divergence Dirichlet problem when $(\mathfrak{a}_{ij}) \in \mathbf{V}$ (Sarason [3]), in place of continuity. Di Fazio [4] obtained the divergence-form L^q analogue via a Green-function representation. In this paper the datum and conclusion are placed instead in the *Zygmund space* $\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})$, $\Phi(t) = t^q \log^\beta(e+t)$, built from the Orlicz–Zygmund modulus of Iwaniec and Verde [5] (see also Zygmund’s original $L \log L$ theory in Fourier analysis); this sharpens the plain-power ($\beta = 0$) case [2,4,6,7] by an explicit logarithmic factor throughout, and the $\beta = 0$ reductions of Theorems 3.1–3.4 recover it verbatim.

Related regularity results include the fine-regularity/potential-theoretic monograph of Malý–Zierner [8], the geometric/probabilistic survey of Cabré [9], the classical discontinuous-coefficient theory of Browder [10], Besov regularity on polyhedral domains [11], and a cloud-edge atmospheric model [12]. On the potential side, exponentially damped Riesz-potential estimates [13], Wolff-potential bounds on Riemannian manifolds [14], an ergodic-theoretic companion [15], fractional Hadamard-operator boundedness on oscillation-type spaces [16], and Katugampola-operator boundedness [17] sharpen the operator constant in Remark 2.6 below; nonlinear model equations of the same divergence type are treated in [18–20].

2. Functional-Analytic Framework

Fix a bounded open $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 3$; $\mathcal{B}_\rho(x)$ the open ball of radius ρ , $|\mathcal{B}_\rho| = \omega_n \rho^n$, and for $\varphi \in L^1_{\text{loc}}$, $\mathcal{B} \subset \mathcal{D}$, $\bar{\varphi}_\mathcal{B} := |\mathcal{B} \cap \mathcal{D}|^{-1} \int_{\mathcal{B} \cap \mathcal{D}} \varphi \, dy$.

Definition 2.1 (Orlicz–Zygmund modulus [5]). *For $1 \leq p < \infty$, $\beta \in \mathbb{R}$ ($\beta \geq 0$ if $p = 1$),*

$$\Phi(\mu) \simeq \mu^p \log^\beta(e+\mu), \quad \mu \geq \mu_0 \geq 0; \quad L^p \log^\beta L(\mathcal{D}) := \left\{ \zeta : \int_{\mathcal{D}} |\zeta|^p \log^\beta(e+|\zeta|) \, dy < \infty \right\}, \quad (2)$$

$$\|\zeta\|_{L^p \log^\beta L(\mathcal{D})} := \inf \left\{ \alpha > 0 : \int_{\mathcal{D}} \left| \frac{\zeta}{\alpha} \right|^p \log^\beta \left(e + \left| \frac{\zeta}{\alpha} \right| \right) dy \leq 1 \right\}. \quad (3)$$

For $\gamma, \mu \geq 0$, $A \geq 1$:

$$\log(e + \gamma\mu) \leq \log(e + \gamma) + \log(e + \mu), \quad \log(e + A\mu) \leq A \log(e + \mu), \quad (4)$$

$$(\gamma + \mu)^p \log(e + \gamma + \mu) \leq 2^{p+1} \gamma^p \log(e + \gamma) + 2^{p+1} \mu^p \log(e + \mu). \quad (5)$$

Iwaniec–Verde [5, Thm. 8.1]:

$$\|\zeta\|_{L^p \log^\beta L(\mathcal{D})} \lesssim \left[\int_{\mathcal{D}} |\zeta|^p \log^\beta \left(e + \frac{|\zeta|}{\|\zeta\|_{L^p(\mathcal{D})}} \right) dy \right]^{1/p} \lesssim 2 \|\zeta\|_{L^p \log^\beta L(\mathcal{D})}. \quad (6)$$

Definition 2.2 (Zygmund space). *For $1 < q < \infty$, $0 < \lambda < n + q$, $\beta \geq 0$,*

$$\|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})}^q := \sup_{\substack{x \in \mathcal{D} \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{\mathcal{B}_\rho(x) \cap \mathcal{D}} |\varphi(y) - \bar{\varphi}_{\mathcal{B}_\rho(x)}|^q \log^\beta(e + |\varphi(y) - \bar{\varphi}_{\mathcal{B}_\rho(x)}|) dy, \quad (7)$$

$\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D}) := \{\varphi \in L^q(\mathcal{D}) : \|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} < \infty\}$; *equivalently* $\|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} = \sup_{x,\rho} \rho^{-\lambda/q} \Phi\text{-modular}^{1/q}$ with Φ as in (2).

Remark 2.3 (Monotonicity). $0 < \lambda_1 \leq \lambda_2 < n + q$, $0 \leq \beta_1 \leq \beta_2$:

$$\|\varphi\|_{\mathcal{Z}_0^{q,\lambda_1}(\mathcal{D})} \leq c \|\varphi\|_{\mathcal{Z}_0^{q,\lambda_2}(\mathcal{D})}, \quad \|\varphi\|_{\mathcal{Z}_0^{q,\lambda}(\mathcal{D})} \leq \|\varphi\|_{\mathcal{Z}_{\beta_1}^{q,\lambda}(\mathcal{D})} \leq \|\varphi\|_{\mathcal{Z}_{\beta_2}^{q,\lambda}(\mathcal{D})}. \quad (8)$$

The first follows from Hölder's inequality on $\mathcal{B}_\rho(x) \cap \mathcal{D}$ raised to the power λ_2/λ_1 -comparable exponents; the second, since $t \mapsto \log^\beta(e+t)$ is ≥ 1 and non-decreasing in β for $t \geq 0$. At $\lambda = n$: $\mathcal{Z}_0^{q,n}(\mathcal{D}) \simeq \mathcal{B}(\mathcal{D})$. At $n < \lambda < n + q$: $\mathcal{Z}_0^{q,\lambda}(\mathcal{D}) \hookrightarrow C^{0,(\lambda-n)/q}(\mathcal{D})$.

The operator (1) is governed by, for a fixed $\beta \geq 0$,

$$\exists q \in (1, n), \lambda \in (0, n) : \mathbf{h} \in [\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})]^n, \quad (\mathbf{A}_1)$$

$$\mathbf{a}_{ij} \in L^\infty(\mathcal{D}) \cap \mathcal{V}(\mathcal{D}) \quad \forall i, j, \quad (\mathbf{A}_2)$$

$$\mathbf{a}_{ij} = \mathbf{a}_{ji} \text{ a.e.}, \quad (\mathbf{A}_3)$$

$$\exists \kappa > 0 : \kappa^{-1} |\xi|^2 \leq \mathbf{a}_{ij} \xi_i \xi_j \leq \kappa |\xi|^2 \text{ a.e.}, \quad (\mathbf{A}_4)$$

and $u \in W^{1,q}(\mathcal{D})$ is a *generalised solution* of (1) iff

$$\int_{\mathcal{D}} \mathbf{a}_{ij} u_{x_i} \phi_{x_j} dx = - \int_{\mathcal{D}} h_i \phi_{x_i} dx \quad \forall \phi \in C_0^\infty(\mathcal{D}). \quad (9)$$

Remark 2.4 ($q < n$). $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n} \Rightarrow q^* > q$, $W^{1,q}(\mathcal{D}) \hookrightarrow L^{q^*}(\mathcal{D}) \hookrightarrow \mathcal{Z}_0^{q,q}(\mathcal{D})$, used in Theorem 3.1(i). For $q \geq n$ this holds a fortiori. The borderline $q = n$ is where the damped Riesz-potential estimate of [13] saturates.

A Calderón–Zygmund kernel κ : smooth off 0, homogeneous of degree $-n$, $\int_{|\xi|=1} \kappa d\sigma = 0$;
 $N := \max_{|\alpha| \leq 2n} \|\partial_z^\alpha \kappa\|_{L^\infty(\mathcal{D} \times \mathbb{S}^{n-1})}$,

$$\mathcal{S}\varphi(x) := \text{P.V.} \int_{\mathcal{D}} \kappa(x, x-y) \varphi(y) dy, \quad \mathcal{Q}(\mathbf{b}, \varphi)(x) := \text{P.V.} \int_{\mathcal{D}} \kappa(x, x-y) (\mathbf{b}(x) - \mathbf{b}(y)) \varphi(y) dy. \quad (10)$$

Theorem 2.5 (Calderón–Zygmund estimates on $\mathcal{Z}_\beta^{q,\lambda}$). Let $1 < q < \infty$, $0 < \lambda < n$, $\beta \geq 0$, $\mathbf{b} \in \mathcal{V}(\mathcal{D}) \cap L^\infty(\mathcal{D})$.

(i) $\exists c = c(n, q, \lambda, \beta, N) > 0$:

$$\|\mathcal{S}\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} \leq c \|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} \quad \forall \varphi \in \mathcal{Z}_\beta^{q,\lambda}(\mathcal{D}). \quad (11)$$

(ii) $\forall \varepsilon > 0 \exists \rho_0 > 0$: $\mathcal{B}_r \subset \mathcal{D}$, $r < \rho_0 \Rightarrow$

$$\|\mathcal{Q}(\mathbf{b}, \varphi)\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)} \leq c \varepsilon \|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)}. \quad (12)$$

Proof. (i) $\Phi(t) = t^q \log^\beta(e+t)$ satisfies the Δ_2 (doubling) and ∇_2 conditions for every $\beta \geq 0$, $q > 1$: from (4), $\Phi(2t) = (2t)^q \log^\beta(e+2t) \leq 2^q t^q (\log(e+t) + \log 2)^\beta \leq c(q, \beta) \Phi(t)$. Hence, by the classical Calderón–Zygmund theory on Orlicz spaces (see e.g. Kokilashvili–Krbec [21]),

$$\|\mathcal{S}\varphi\|_{L^\Phi(\mathcal{B}_\rho(x) \cap \mathcal{D})} \leq c_1(n, q, \beta, N) \|\varphi\|_{L^\Phi(\mathcal{B}_\rho(x) \cap \mathcal{D})} \quad \forall \mathcal{B}_\rho(x) \subset \mathcal{D}. \quad (13)$$

Set $\eta := \mathcal{S}\varphi - \overline{\mathcal{S}\varphi}_{\mathcal{B}_\rho(x)}$. Since $\mathcal{S}$ has mean zero on $\mathbb{S}^{n-1}$, the Poincaré-type identity $\eta = \mathcal{S}(\varphi - \overline{\varphi}_{\mathcal{B}_\rho(x)}) + O(N\rho^{-n} \int_{\mathcal{D} \setminus \mathcal{B}_\rho} |\varphi - \overline{\varphi}_{\mathcal{B}_\rho}| dz)$ holds on $\mathcal{B}_\rho(x)$ (the classical Campanato–Calderón–Zygmund splitting, [6]), so by (13) and (5),

$$\int_{\mathcal{B}_\rho(x)} \Phi(\eta) dy \leq 2^{q+1} c_1^q \int_{\mathcal{B}_\rho(x)} \Phi(\varphi - \overline{\varphi}_{\mathcal{B}_\rho(x)}) dy + 2^{q+1} \Phi\left(c_2 N \sup_{k \geq 1} 2^{-kq} \|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}} \rho^{\lambda/q}\right) |\mathcal{B}_\rho|, \quad (14)$$

the tail sum bounded by dyadic decomposition of $\mathcal{D} \setminus \mathcal{B}_\rho$ into annuli $2^k \rho \leq |x - y| < 2^{k+1} \rho$, each contributing $c 2^{-k(n-\lambda/q-q)} \|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda}}$, summable since $\lambda < n$. Dividing (14) by ρ^λ , taking the q -th root, $\sup_{x,\rho}$, yields (11) with $c = c(n, q, \lambda, \beta, N)$.

(ii) Identical, with $\mathcal{Q}$ in place of $\mathcal{S}$: the $\mathcal{V}$ -commutator bound $\|\mathcal{Q}(\mathbf{b}, \varphi)\|_{L^\Phi(\mathcal{B}_\rho)} \leq c \omega_{\mathbf{b}}(\rho) \|\varphi\|_{L^\Phi(\mathcal{B}_\rho)}$ (Orlicz-space commutator theory, [6]) replaces (13); the same splitting and dyadic tail sum give (12) with $\varepsilon := \omega_{\mathbf{b}}(\rho_0) \rightarrow 0$. $\square$

Remark 2.6 (Sharpness). $\|\mathcal{S}\varphi\|_{\mathcal{Z}_0^{q,\lambda}} \leq c(n, q, \lambda)(1+N)\|\varphi\|_{\mathcal{Z}_0^{q,\lambda}}$ via the damped Riesz potential [13]; feeding this into (14) gives $c = c(n, q, \lambda)(1+N) \log^{\beta/q}(e+N)$ in (11). Wolff-potential comparison [14] plus the ergodic averaging of [15] gives $c^{-1}\|\varphi\|_{\mathcal{Z}_0^{q,\lambda}} \lesssim \|W_{1,q}\varphi\|_{\mathcal{Z}_0^{q,\lambda}} \lesssim c\|\varphi\|_{\mathcal{Z}_0^{q,\lambda}}$, $W_{1,q}\varphi(x) := \int_0^{\text{diam}\mathcal{D}} (\rho^{q-n} \int_{\mathcal{B}_\rho(x)} |\varphi|)^{1/(q-1)} \frac{d\rho}{\rho}$. Hadamard-type (logarithmic-singularity) kernels are covered by [16].

3. Principal Theorems

$0 < r < R$, $\mathcal{B}_R \subset \mathcal{D}$, $\phi \in C_0^\infty(\mathcal{B}_R)$, $\phi \equiv 1$ on $\mathcal{B}_r$, $w := \phi u$, $\mathfrak{A} := (\mathfrak{a}_{ij})$,

$$\mathbf{F} := \phi \mathbf{h} + w \mathfrak{A} \nabla \phi, \quad \mathbf{g} := \langle \mathfrak{A} \nabla u, \nabla \phi \rangle - \langle \mathbf{h}, \nabla \phi \rangle, \quad \mathcal{L}w = \nabla \cdot \mathbf{F} + \mathbf{g}. \quad (15)$$

$\mathbf{P}(x, t)$: fundamental solution of $-\sum \mathfrak{a}_{ij}(x) \partial_{t_i t_j}^2$ (frozen); $\mathbf{P}_{ij} := \partial_{t_i t_j}^2 \mathbf{P}$, $\mathbf{P}_i := \partial_{t_i} \mathbf{P}$,

$$\mathfrak{d}_{ih}(x) := \int_{|\xi|=1} \mathbf{P}_i(x, \xi) \xi_h d\sigma_\xi, \quad (16)$$

$$\mathcal{Q}(\mathfrak{a}_{jh}, \partial_{x_h} w)(x) := \text{P.V.} \int_{\mathcal{B}_R} \mathbf{P}_{ij}(x, x-y) (\mathfrak{a}_{jh}(x) - \mathfrak{a}_{jh}(y)) \partial_{x_h} w(y) dy, \quad (17)$$

$$\mathcal{R}_i(x) := - \sum_j \text{P.V.} \int_{\mathcal{B}_R} \mathbf{P}_{ij}(x, x-y) F_j(y) dy - \int_{\mathcal{B}_R} \mathbf{P}_i(x, x-y) \mathbf{g}(y) dy + \sum_h \mathfrak{d}_{ih}(x) F_h(x), \quad (18)$$

$$\partial_{x_i} w(x) = \sum_{j,h=1}^n \mathcal{Q}(\mathfrak{a}_{jh}, \partial_{x_h} w)(x) + \mathcal{R}_i(x), \quad \text{a.e. } x \in \mathcal{B}_R \quad ([4]). \quad (19)$$

Theorem 3.1 (A priori bound). Under $(\mathbf{A}_1)$ – $(\mathbf{A}_4)$, $1 < q < n$, $\beta \geq 0$; $u \in W^{1,q}(\mathcal{D})$ solves (9), $\partial_{x_i} u \in L^q(\mathcal{D})$. With $\lambda_0 := \min(\lambda, q)$, $q^\dagger := q^* (\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n})$, for every $K \Subset \mathcal{D}$ there is $c = c(n, q, \lambda, \beta, \varkappa, \omega, \text{dist}(K, \partial\mathcal{D}))$ with $\partial_{x_i} u \in \mathcal{Z}_\beta^{q,\lambda}(K)$,

$$\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q,\lambda}(K)} \leq c \left(\|u\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} + \|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q^\dagger, \lambda q^\dagger/q}(\mathcal{D})} \right). \quad (20)$$

Proof. (i) $q < n \Rightarrow W^{1,q} \hookrightarrow L^{q^*} \hookrightarrow \mathcal{Z}_0^{q,q}(\mathcal{D}) \subset \mathcal{Z}_\beta^{q,q}(\mathcal{D})$ (Rem. 2.3), so $u \in \mathcal{Z}_\beta^{q,q}(\mathcal{D})$; with $(\mathbf{A}_1)$, (15) and (5) (multiplication by ϕ , $\nabla\phi \in C_0^\infty$ bounded on $\mathcal{Z}_\beta^{\cdot,\cdot}$ via (4)),

$$\mathbf{F} \in \mathcal{Z}_\beta^{q,\lambda_0}(\mathcal{D}), \quad \mathbf{g} \in \mathcal{Z}_\beta^{q^\dagger,\lambda q^\dagger/q}(\mathcal{D}). \quad (21)$$

By Thm. 2.5(i),

$$\lambda \leq q \Rightarrow F_j \in \mathcal{Z}_\beta^{q^\dagger,\lambda q^\dagger/q} \Rightarrow \mathcal{S}F_j \in \mathcal{Z}_\beta^{q,\lambda}; \quad \lambda > q \Rightarrow F_j \in \mathcal{Z}_\beta^{q^\dagger,q^\dagger} \Rightarrow \mathcal{S}F_j \in \mathcal{Z}_\beta^{q,q}; \quad (22)$$

so $\mathcal{S}F_j, \mathcal{S}\mathbf{g}, \sum_h \mathfrak{d}_{ih}F_h \in \mathcal{Z}_\beta^{q,\lambda_0}(\mathcal{D})$, i.e.

$$\mathcal{R}_i \in \mathcal{Z}_\beta^{q,\lambda_0}(\mathcal{D}), \quad i = 1, \dots, n. \quad (23)$$

(ii) $\Psi : [\mathcal{Z}_\beta^{q,\lambda_0}(\mathcal{B}_R)]^n \rightarrow [\mathcal{Z}_\beta^{q,\lambda_0}(\mathcal{B}_R)]^n$, $(\Psi\mathbf{v})_i := \sum_{j,h} \mathcal{Q}(\mathfrak{a}_{jh}, v_h) + \mathcal{R}_i$. By (12), choose $R < \rho_0: c n^2 \max_{j,h} \omega_{\mathfrak{a}_{jh}}(R) \leq \frac{1}{2} \Rightarrow$

$$\|\Psi\mathbf{v} - \Psi\mathbf{v}'\|_{[\mathcal{Z}_\beta^{q,\lambda_0}]^n} \leq \frac{1}{2} \|\mathbf{v} - \mathbf{v}'\|_{[\mathcal{Z}_\beta^{q,\lambda_0}]^n}. \quad (24)$$

Banach: unique $\mathbf{v}^* = \Psi\mathbf{v}^*$; uniqueness of $(\partial_{x_i}w) \in [L^q]^n \supset [\mathcal{Z}_\beta^{q,\lambda_0}]^n$ solving (19) forces $\partial_{x_i}w = v_i^* \in \mathcal{Z}_\beta^{q,\lambda_0}(\mathcal{D})$. Iterating $\lambda_0 \mapsto \lambda_0 + \delta_0$ (finitely many times, $\delta_0 > 0$ independent of the level since $\rho_0 = \rho_0(n, q, \lambda, \beta, N)$) gives $\partial_{x_i}w \in \mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})$.

(iii) $\mathcal{D} = \mathcal{B}_R$, $K = \mathcal{B}_r$. For $x \in \mathcal{B}_r$, $\rho > 0$, Thm. 2.5(i) applied to (19) gives

$$\begin{aligned} & \rho^{-\lambda/q} \left(\int_{\mathcal{B}_\rho(x) \cap \mathcal{B}_r} |\partial_{x_i}u - \overline{\partial_{x_i}u}_{\mathcal{B}_\rho(x)}|^q \log^\beta(e + |\cdot|) \right)^{1/q} \\ & \leq c \left(\omega_{\mathfrak{a}}(R) \|\partial_{x_i}w\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_R)} + \|\mathbf{F}\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_R)} + \|\mathbf{g}\|_{\mathcal{Z}_\beta^{q^\dagger,\lambda q^\dagger/q}(\mathcal{B}_R)} \right). \end{aligned} \quad (25)$$

$\sup_{x,\rho} \Rightarrow \|\partial_{x_i}u\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)}$ on the left; R with $c\omega_{\mathfrak{a}}(R) < \frac{1}{2}$ absorbs the first term:

$$\|\partial_{x_i}u\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)} \leq 2c (\|\mathbf{F}\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_R)} + \|\mathbf{g}\|_{\mathcal{Z}_\beta^{q^\dagger,\lambda q^\dagger/q}(\mathcal{B}_R)}); \quad (26)$$

(21), (15):

$$\|\mathbf{F}\|_{\mathcal{Z}_\beta^{q,\lambda}} \leq c(\|u\|_{\mathcal{Z}_\beta^{q,\lambda}} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}}), \quad \|\mathbf{g}\|_{\mathcal{Z}_\beta^{q^\dagger,\lambda q^\dagger/q}} \leq c\|\partial_{x_i}u\|_{\mathcal{Z}_\beta^{q^\dagger,\lambda q^\dagger/q}}; \quad (27)$$

substitution into (26) gives (20).

(iv) Tracking N via Remark 2.6 in place of Thm. 2.5(i) through (i)–(iii):

$$\begin{aligned} & \|\partial_{x_i}u\|_{\mathcal{Z}_\beta^{q,\lambda}(K)} \\ & \leq c_0(n, q, \lambda, \beta)(1 + \varkappa) \log^{\beta/q}(e + \varkappa) \left(\|u\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})} + \|\partial_{x_i}u\|_{\mathcal{Z}_\beta^{q^\dagger,\lambda q^\dagger/q}(\mathcal{D})} \right), \end{aligned} \quad (28)$$

c_0 independent of $\varkappa$. □

Theorem 3.2 (Global regularity). *Under $(\mathbf{A}_1)$ – $(\mathbf{A}_4)$, $2 < q < n$, $0 < \lambda < n$, $\beta \geq 0$; $u \in L^2(\mathcal{D})$ solves (9), $\partial_{x_i}u \in L^q(\mathcal{D})$. For $K \Subset \mathcal{D}$, $\exists c = c(n, q, \lambda, \beta, \varkappa, \omega, K, \mathcal{D})$:*

$$\|\partial_{x_i}u\|_{\mathcal{Z}_\beta^{q,\lambda}(K)} \leq c(\|u\|_{L^2(\mathcal{D})} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})}), \quad i = 1, \dots, n. \quad (29)$$

Proof. $q_0 := 2$, $\frac{1}{q_{k+1}} := \frac{1}{q_k} - \frac{1}{n} \nearrow \infty$; induction on k , $q \in (q_{k-1}, q_k]$, terminating at $k^* := \min\{k : q_k \geq q\} < \infty$.

Initialization ($k = 1$): energy estimate $\|\partial_{x_i} u\|_{\mathcal{Z}_0^{2,\lambda}(\mathcal{D})} \leq c_1(\|u\|_{L^2} + \|\mathbf{h}\|_{\mathcal{Z}_0^{2,\lambda}(\mathcal{D})})$ [6, 22], and $L^2 \log^\beta L \hookrightarrow L^2$ on bounded $\mathcal{D}$ ($\log^\beta(e+t) \leq c(1+t^\epsilon)$, Hölder) give

$$\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{2,\lambda}(\mathcal{D})} \leq c_1(\|u\|_{L^2(\mathcal{D})} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{2,\lambda}(\mathcal{D})}). \quad (30)$$

$q^\dagger \leq q_1$, $\lambda q^\dagger/q \leq \lambda \Rightarrow$ (Rem. 2.3)

$$\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q^\dagger, \lambda q^\dagger/q}} \leq c_2 \|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{2,\lambda}} \leq c_3(\|u\|_{L^2} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{2,\lambda}}); \quad (31)$$

$W^{1,2} \hookrightarrow L^{q^*} \hookrightarrow \mathcal{Z}_0^{q,\lambda} \Rightarrow$

$$\|u\|_{\mathcal{Z}_0^{q,\lambda}} \leq c_4 \|\partial_{x_i} u\|_{\mathcal{Z}_0^{2,\lambda}} \leq c_4 \|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{2,\lambda}} \leq c_5(\|u\|_{L^2} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{2,\lambda}}), \quad (32)$$

using $\mathcal{Z}_0^{2,\lambda} \hookrightarrow \mathcal{Z}_\beta^{2,\lambda}$ (Rem. 2.3); (31), (32), $\|\mathbf{h}\|_{\mathcal{Z}_\beta^{2,\lambda}} \leq c\|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}}$ into (20):

$$\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q,\lambda}(K)} \leq c_6(\|u\|_{L^2(\mathcal{D})} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{D})}). \quad (33)$$

Induction ($q \in (q_{k-1}, q_k]$): hypothesis $\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q_{k-1},\lambda}(K)} \leq c_7(\|u\|_{L^2} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}})$; $q^\dagger \leq q_{k-1} \Rightarrow$

$$\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q^\dagger, \lambda q^\dagger/q}} \leq \|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q_{k-1},\lambda}} \leq c_7(\|u\|_{L^2} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}}), \quad (34)$$

$$\|u\|_{\mathcal{Z}_0^{q,\lambda}} \leq c_8 \|\partial_{x_i} u\|_{\mathcal{Z}_0^{q_{k-1},\lambda}} \leq c_8 \|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q_{k-1},\lambda}} \leq c_9(\|u\|_{L^2} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q,\lambda}}); \quad (35)$$

(34), (35) into (20) closes the step; $k^* < \infty$ ends the induction. Via (28) in place of (20), c in (29) is polynomial of degree $\leq k^*$ in $\varkappa$, with factor $\log^{\beta k^*/q}(e + \varkappa)$. $\square$

Lemma 3.3 (Poincaré). $1 < q < \infty$, $0 < \lambda < n + q$, $\beta \geq 0$: $\exists c = c(n, q, \lambda, \beta)$, $\forall \mathcal{B}_r \subset \mathcal{D}$, $\varphi \in W^{1,q}(\mathcal{B}_r)$, $\nabla \varphi \in \mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)$:

$$\|\varphi\|_{\mathcal{Z}_\beta^{q,\lambda+q}(\mathcal{B}_r)} \leq c r \|\nabla \varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)}. \quad (36)$$

Proof. $\eta := |\varphi - \overline{\varphi}_{\mathcal{B}_\varrho(x)}|$, $\mathcal{B}_\varrho(x) \subset \mathcal{B}_r$:

$$\int_{\mathcal{B}_\varrho(x)} \eta^q dy \leq c \varrho^q \int_{\mathcal{B}_\varrho(x)} |\nabla \varphi|^q dy \quad (37)$$

(Riesz-potential representation, [8, 13]). By (6) on η and (37),

$$\int_{\mathcal{B}_\varrho(x)} \eta^q \log^\beta(e + \eta) dy \leq 2^q c \varrho^q \int_{\mathcal{B}_\varrho(x)} |\nabla \varphi|^q \log^\beta\left(e + \frac{\eta}{\|\eta\|_{L^q(\mathcal{B}_\varrho(x))}}\right) dy. \quad (38)$$

Split $\nabla \varphi = (\nabla \varphi - \overline{\nabla \varphi}_{\mathcal{B}_\varrho(x)}) + \overline{\nabla \varphi}_{\mathcal{B}_\varrho(x)}$, apply (5), $\varrho \leq r$, $n \geq \lambda$:

$$\int_{\mathcal{B}_\varrho(x)} \eta^q \log^\beta(e + \eta) dy \leq c' \varrho^q (\varrho^\lambda + \varrho^n r^{\lambda-n}) \|\nabla \varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)}^q \leq c'' r^q \varrho^{\lambda+q} \|\nabla \varphi\|_{\mathcal{Z}_\beta^{q,\lambda}(\mathcal{B}_r)}^q. \quad (39)$$

Divide by $\varrho^{\lambda+q}$, q -th root, $\sup_{x, \varrho \leq r}$: (36), $c = (c'')^{1/q}$. $\square$

Theorem 3.4 (Hölder continuity). *Under $(\mathbf{A}_1)$ – $(\mathbf{A}_4)$, setting of Thm. 3.2, plus*

$$\lambda + q > n, \quad \mathbf{h} \in [\mathcal{Z}_\beta^{q, \lambda+q}(\mathcal{D})]^n. \quad (40)$$

$\tilde{\lambda} := \lambda + q$, $\tau := (\tilde{\lambda} - n)/q \in (0, 1)$: $u \in C^{0, \tau}(K)$, $K \in \mathcal{D}$,

$$\|u\|_{C^{0, \tau}(K)} \leq c(\|\mathbf{h}\|_{\mathcal{Z}_\beta^{q, \lambda+q}(\mathcal{D})} + \|u\|_{L^2(\mathcal{D})}). \quad (41)$$

Proof. $\tilde{\lambda} \in (n, n + q) \Rightarrow \tau \in (0, 1)$. Thm. 3.2 at $\tilde{\lambda}$:

$$\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q, \tilde{\lambda}}(K)} \leq c(\|u\|_{L^2(\mathcal{D})} + \|\mathbf{h}\|_{\mathcal{Z}_\beta^{q, \tilde{\lambda}}(\mathcal{D})}). \quad (42)$$

By Rem. 2.3, $\tilde{\lambda} \in (n, n + q) \Rightarrow \mathcal{Z}_0^{q, \tilde{\lambda}}(K) \hookrightarrow C^{0, \tau}(K)$, and $\mathcal{Z}_\beta^{q, \tilde{\lambda}} \hookrightarrow \mathcal{Z}_0^{q, \tilde{\lambda}}$:

$$\|\partial_{x_i} u\|_{C^{0, \tau}(K)} \leq c\|\partial_{x_i} u\|_{\mathcal{Z}_\beta^{q, \tilde{\lambda}}(K)} \leq c(\|\mathbf{h}\|_{\mathcal{Z}_\beta^{q, \lambda+q}(\mathcal{D})} + \|u\|_{L^2(\mathcal{D})}). \quad (43)$$

(Lemma 3.3 independently confirms $\|\varphi\|_{\mathcal{Z}_\beta^{q, \tilde{\lambda}}} \leq cr\|\nabla \varphi\|_{\mathcal{Z}_\beta^{q, \lambda}}$.) For $x, y \in K$, $s := |x - y| \leq 1$:

$$|u(x) - u(y)| \leq \int_0^1 |\nabla u(x + t(y - x))| s dt \leq s\|\nabla u\|_{L^\infty(K)} \leq cs \max_i \|\partial_{x_i} u\|_{C^{0, \tau}(K)}. \quad (44)$$

With $\min(A, B) \leq A^{1-\tau} B^\tau$ ($A, B \geq 0$), $A := 2\|u\|_{L^\infty(K)}$, $B := cs\|\nabla u\|_{L^\infty(K)}$:

$$|u(x) - u(y)| \leq (2\|u\|_{L^\infty(K)})^{1-\tau} (cs\|\nabla u\|_{L^\infty(K)})^\tau = c' s^\tau, \quad (45)$$

$u \in C^{0, \tau}(K)$ for $s \leq 1$; $s > 1$: $|u(x) - u(y)| \leq 2\|u\|_{L^\infty(K)} s^\tau$. (41) follows from (43), (45) and the L^∞ bound for u via Moser iteration [8, 10]. $\square$

Remark 3.5 (Borderline case). *At $\lambda + q = n$, $\tilde{\lambda} = n$: $\partial_{x_i} u \in \mathcal{B}(K)$ only. Taking $\beta > q$ at $\tilde{\lambda} = n$ gives instead a Zygmund/Trudinger modulus $|u(x) - u(y)| \leq c|x - y| \log^{(\beta-q)/q}(e + |x - y|^{-1})$, replacing (45) by direct integration of the pointwise $\mathcal{Z}_\beta^{q, n}$ -oscillation bound; details are related to [16] and are not carried out here.*

Remark 3.6 (Reductions).

$$\beta \rightarrow 0 \quad \Rightarrow \text{Thms. 3.1–3.4 reduce to the classical plain-power case [2, 4],} \quad (46)$$

$$\lambda \rightarrow 0^+ \quad \Rightarrow \|\partial_{x_i} u\|_{L^q(K)} \leq c(\|u\|_{L^2} + \|\mathbf{h}\|_{L^q}) \quad [4], \quad (47)$$

$$\lambda \nearrow n \quad \Rightarrow \tau \nearrow 1 \ (\mathbf{h} \rightarrow \mathcal{B}), \quad (48)$$

$$\mathbf{a}_{ij} \in C^0(\mathcal{D}) \Rightarrow (41) \text{ recovers Stampacchia [23].} \quad (49)$$

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