



Research Article

Fejér-Type Inequalities via Generalized Superquadraticity with Application to Cumulative Fatigue Damage

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Abstract

This paper introduces a new class of superquadratic functions, termed s-superquadratic functions. Based on this new definition, two fundamental lemmas are established, forming the basis for the derivation of several refined Fejér-type integral inequalities. Furthermore, the theoretical findings are applied to fatigue mechanics, where the derived inequalities furnish rigorous lower and upper bounds for cumulative fatigue damage under cyclic loading. The proposed estimates incorporate the nonlinear evolution of fatigue damage through the s-superquadratic framework, providing guaranteed bounds for the accumulated fatigue damage based on weighted loading distributions and cycle-dependent damage densities.

1. Introduction

Fatigue is one of the most pervasive failure mechanisms in engineering materials and structural components subjected to repeated or cyclic loading. The process of predicting when a component will fail relies heavily on the concept of cumulative fatigue damage. Traditionally, linear damage accumulation models, such as the Palmgren-Miner rule, have been utilized to sum discrete damage increments. However, real-world structural degradation often exhibits highly nonlinear, cycle-dependent behavior (e.g., Marco-Starkey models). Because evaluating exact nonlinear damage integrals over complex, randomized load histories is computationally and analytically demanding, finding rigorous mathematical boundaries—specifically upper and lower estimates, for cumulative damage is of immense engineering value.

To develop these bounds, we turn to the theory of convex analysis and integral inequalities. Let the mapping $\Phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex on I , and let $\varepsilon_1, \varepsilon_2 \in I$ with $\varepsilon_1 < \varepsilon_2$. Among the most fundamental and celebrated results in the theory of convexity is the **H-H** inequality, which establishes sharp lower and upper bounds for the integral mean of a convex mapping in terms of its midpoint value and the arithmetic mean of its endpoint values. This classical double inequality is given by [1]:

$$\Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \leq \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \Phi(\mathfrak{w}) d\mathfrak{w} \leq \frac{\Phi(\varepsilon_2) + \Phi(\varepsilon_1)}{2}. \quad (1)$$

For the literature, see [2, 3, 4, 5].

In the context of convex analysis, the **H-H** inequality is a pivotal result for estimating the integral mean of a convex function on an interval, ensuring that the function is Riemann integrable over compact intervals. More precisely, one is able to derive numerous classical mean inequalities in a direct way through the **H-H** inequality, provided one picks appropriate convex functions Φ . The significance of the **H-H** inequality in terms of both theory and applications makes it an indispensable instrument of modern mathematical analysis that finds its use in optimization, approximation theory, numerical integration, probability and statistics, fractional calculus, information theory, and many other fields.

In the contemporary literature on integral inequalities, the deviation in the second part of inequality (1) (the difference between the integral mean and the trapezoidal approximation using endpoints) is known as the trapezoid-type inequality, first developed by Dragomir and his colleague in 1998 [6]. Subsequently, Kirmaci formulated the corresponding inequality for the first part of inequality (1) (the deviation between the integral mean and the midpoint value), which is now widely recognized as the midpoint inequality [7]. These two inequalities, the trapezoid and midpoint inequalities have played a pivotal role in the development of the theory of integral inequalities. They serve as indispensable analytical tools for deriving sharper estimates, establishing refined error bounds, and they have inspired numerous extensions, refinements, and generalizations under a broad spectrum of generalized convexity assumptions.

To set the stage for our generalizations, we briefly review the geometric interpretation of convexity that underpins the idea of superquadracity. Suppose that $\Phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a real-valued function, and $(\varepsilon_1, \Phi(\varepsilon_1)), (\varepsilon_2, \Phi(\varepsilon_2)) \in I$ are two arbitrary points of its graph, where $\varepsilon_1 < \varepsilon_2$. Geometrically, convexity means that the graph of Φ lies below or on the straight line (chord) connecting the two points. Equivalently, any point of the graph corresponding to a convex combination of ε_1 and ε_2 does not exceed the corresponding combination of their function values.

In analytical terms, convexity is characterized by the fact that for all $\mathfrak{w}$ with $\varepsilon_1 \leq \mathfrak{w} \leq \varepsilon_2$,

$$\Phi(\mathfrak{w}) \leq \frac{\Phi(\varepsilon_2) - \Phi(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1}(\mathfrak{w} - \varepsilon_1) + \Phi(\varepsilon_1),$$

Another characterization of convexity is given by the existence of supporting lines. Specifically, a function $\Phi : I \rightarrow \mathbb{R}$ is said to admit a supporting line at a point $\varepsilon_1 \in I$, if there exists a

$$\Phi(\mathfrak{w}) \geq \Phi(\varepsilon_1) + \mathfrak{C}_{\varepsilon_1}(\mathfrak{w} - \varepsilon_1), \quad (2)$$

for each $\mathfrak{w} \in I$. The set of these constants $\mathfrak{C}_{\varepsilon_1}$ is called the subdifferential of Φ at ε_1 and is denoted by $\partial\Phi(\varepsilon_1)$. Geometrically, the subdifferential represents the family of slopes of all lines that support Φ from below at $(\varepsilon_1, \Phi(\varepsilon_1))$. Consequently, the subdifferential provides a comprehensive local characterization of the convexity of Φ , reflecting the admissible linear approximations from below. In particular, a fundamental result in convex analysis asserts that if Φ is convex and ε_1 is an interior point of I , then $\partial\Phi(\varepsilon_1)$ is non-empty, guaranteeing at least one supporting line at each interior point. This constitutes one of the most fundamental geometric and analytical characterizations of convex functions.

Building on this perspective, Abramovich et al. [8] introduced the concept of superquadratic functions as a significant extension of classical convexity. This notion broadens convex analysis by considering a larger class of functions that satisfy an even stronger inequality than (2), thereby allowing derivation of sharper inequalities than those obtained from traditional convexity. We recall the formal definition of superquadratic functions and some of their properties below.

Definition 1.1. A mapping $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is said to be superquadratic if, for each $\varepsilon_1 \geq 0$, there is constant $\mathfrak{C}_{\varepsilon_1} \in \mathbb{R}$ so that the following result is satisfied. This characterization extends the classical supporting-line property of convex functions by incorporating an additional quadratic deviation term, thereby providing a substantially richer geometric and analytical framework for the investigation of generalized convexity and its associated integral inequalities:

$$\Phi(\mathfrak{w}) \geq \Phi(\varepsilon_1) + \mathfrak{C}_{\varepsilon_1}(\mathfrak{w} - \varepsilon_1) + \Phi(|\mathfrak{w} - \varepsilon_1|), \quad (3)$$

holds for every $\mathfrak{w} \geq 0$. The superquadraticity condition poses a much stronger geometric restriction on the function when compared to the one of convexity in the classical sense of a supporting line definition. More specifically, the function satisfying the condition of being superquadratic is supposed to be situated strictly above the supporting affine function with a shift of the function itself added to it. This new term adds much geometrical information to the definition of the function thus giving us more analytical properties and sharper integral inequalities as compared to the convexity case.

Remark 1.2. If Φ is superquadratic, then its additive inverse $-\Phi$ is called subquadratic. Thus, the inequality (3) holds in the reversed direction. In other words, every result obtained for superquadratic functions has a corresponding dual result for subquadratic functions, yielding a parallel theory that significantly extends the scope of quadratic inequalities.

The notion of superquadratic functions and their fundamental properties were first introduced by Abramovich et al. in [8, 9]. The condition (3) is a much stronger requirement than classical convexity, making superquadratic functions especially useful for deriving generalized convexity results and integral inequalities. It should be noted, however, that the superquadraticity condition is most effective when Φ is non-negative. If Φ takes negative values, the strength of condition (3) is greatly diminished, and it may not yield significant advantages in analysis. Nevertheless,

Abramovich et al. [8] showed that any non-negative superquadratic function (with $\Phi(0) = \Phi'(0) = 0$) is necessarily convex and increasing, highlighting the strong link between non-negative, convexity, and superquadraticity. The next lemma summarizes some basic properties of superquadratic functions.

Lemma 1.3. [8] *Let the map Φ be superquadratic, then*

1. $\Phi(0) \leq 0$.
2. *If Φ is differentiable at $w > 0$ and $\Phi(0) = \Phi'(0) = 0$ then $\Phi'(w) = \mathfrak{C}_w$.*
3. Φ is convex if $\Phi \geq 0$ and $\Phi(0) = \Phi'(0) = 0$.

This lemma provides conditions under which classical convexity or concavity imply superquadratic or subquadratic behavior. In particular, a convex function that is non-negative and satisfies $\Phi(0) = \Phi'(0) = 0$ will meet the superquadratic condition (4), and similarly a concave function with $\Phi(0) = \Phi'(0) = 0$ (and non-positive values) will be subquadratic. In this way, superquadraticity and subquadraticity can be seen as strengthened forms of convexity and concavity under these conditions. A further important characterization of superquadratic functions is given by a generalized Jensen's inequality. The following definition, adapted from [9], provides a Jensen-type criterion for superquadraticity.

Definition 1.4. *A mapping $\Phi : [0, \infty) \rightarrow R$ is said to be superquadratic, if*

$$\begin{aligned} \Phi((1-\varrho)\varepsilon_1 + \varrho\varepsilon_2) &\leq (1-\varrho)[\Phi(\varepsilon_1) - \Phi(\varrho|\varepsilon_1 - \varepsilon_2|)] \\ &+ \varrho[\Phi(\varepsilon_1) - \Phi((1-\varrho)|\varepsilon_1 - \varepsilon_2|)] \end{aligned} \quad (4)$$

holds, $\forall \varepsilon_2, \varepsilon_1 \geq 0$ and $0 \leq \varrho \leq 1$.

Over the years, several generalizations of superquadratic functions have been proposed. For example, Alomari and Chesneau introduced the concept of h-superquadratic functions as a natural extension of the classical superquadratic notion [10]. Another interesting generalization is the class of multiplicatively superquadratic functions, defined by Krnić et al. is often referred to as the class of logarithmically superquadratic functions [11]. Butt and Khan applied this multiplicative framework to derive **H-H**-type inequalities via multiplicative calculus [12]. Khan and Butt developed a systematic study of P-superquadraticity, obtaining a variety of sharp integer-order and fractional-order inequalities with practical applications [13]. This line of inquiry was further expanded by Khan et al., who introduced the more general notion of (P, m)-superquadraticity, broadening the superquadratic framework with additional parameters and demonstrating new properties and integral inequalities [14]. More recently, Butt and his co-author proposed a novel extension of superquadraticity to interval-valued functions, establishing corresponding integral inequalities and highlighting notable applications in information theory via interval fractional calculus [15]. These developments deepen the connection between generalized convexity, multiplicative calculus, interval analysis, fractional calculus, and information theory, opening new avenues for future research in these connected areas. For a comprehensive overview of the theories of convexity and superquadraticity and their diverse applications, we refer the reader to [16, 17, 18] and references therein.

The main contribution of this work is the introduction of a new class of generalized convex functions, namely s-superquadratic functions, which extends the concept of superquadraticity by incorporating a parameter $s \in (0, 1]$. Based on this framework, we establish refined Fejér-type weighted integral inequalities that not only recover many known results as special cases,

but also provide significant extensions and sharper bounds for classical integral inequalities. Moreover, we apply the theoretical results to fatigue mechanics, where the derived inequalities yield rigorous lower and upper bounds for cumulative fatigue damage under cyclic loading. These bounds capture the nonlinear accumulation of fatigue damage through the s -superquadratic framework, demonstrating the practical utility of the proposed theory for fatigue-life prediction, cumulative damage assessment, structural health monitoring, reliability analysis, and the design of engineering components subjected to repeated loading.

2. Fejér-type integral inequalities via s -superquadraticity

In this section, we present the principal findings of the paper concerning the newly introduced class of s -superquadratic function.

Definition 2.1. A function $\Phi : I \rightarrow (0, \infty)$, is said to be s -superquadratic, if the below mentioned condition

$$\begin{aligned} & \Phi(\varrho v + (1 - \varrho)\vartheta) \\ & \leq \varrho^s [\Phi(v) - \Phi((1 - \varrho)|v - \vartheta|)] + (1 - \varrho)^s [\Phi(\vartheta) - \Phi(\varrho|v - \vartheta|)], \end{aligned} \quad (5)$$

holds for every $v, \vartheta \in I$, $\varrho \in [0, 1]$ and $s \in (0, 1]$.

Remark 2.2. If the relation symbol in (5) is reversed, then the process $\Phi(\mathcal{Z}, \cdot)$ is said to be a s -subquadratic function. Similarly by choosing $s = 1$, the notion of an s -superquadratic function coincides with that of a classical superquadratic function.

Lemma 2.3. Let the function $\Phi : I \rightarrow (0, \infty)$ be a s -superquadratic, then

$$\frac{\Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)}{\left(\frac{1}{2}\right)^s} \leq \Phi(\varkappa) + \Phi(\varepsilon_1 + \varepsilon_2 - \varkappa) - 2\Phi\left(\left|\frac{\varepsilon_1 + \varepsilon_2}{2} - \varkappa\right|\right),$$

for all $\varepsilon_1, \varepsilon_2 \in I$ and $s \in (0, 1]$.

Proof. Applying the s -superquadraticity of Φ with $v = \varkappa$, $\vartheta = \varepsilon_1 + \varepsilon_2 - \varkappa$, $\varrho = \frac{1}{2}$, we obtain

$$\Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \leq \left(\frac{1}{2}\right)^s \left[\Phi(\varkappa) + \Phi(\varepsilon_1 + \varepsilon_2 - \varkappa) - 2\Phi\left(\left|\frac{\varepsilon_1 + \varepsilon_2}{2} - \varkappa\right|\right) \right].$$

Dividing by $\left(\frac{1}{2}\right)^s$ gives the desired result. □

Lemma 2.4. Let the function $\Phi : I \rightarrow (0, \infty)$ be an s -superquadratic. Then

$$\begin{aligned} & \Phi(\varkappa) + \Phi(\varepsilon_1 + \varepsilon_2 - \varkappa) \leq (\Phi(\varepsilon_1) + \Phi(\varepsilon_2)) \left[\left(\frac{\varkappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s + \left(\frac{\varepsilon_2 - \varkappa}{\varepsilon_2 - \varepsilon_1}\right)^s \right] \\ & - 2 \left[\left(\frac{\varkappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\varepsilon_2 - \varkappa|) + \left(\frac{\varepsilon_2 - \varkappa}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\varkappa - \varepsilon_1|) \right], \end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 \in I$ and $s \in (0, 1]$.

Proof. Applying s -superquadraticity of Φ , with $v = \varepsilon_1$, $\vartheta = \varepsilon_2$, $\varrho = \frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1}$, gives $\varrho v + (1 - \varrho)\vartheta = \kappa$, and hence

$$\begin{aligned}\Phi(\kappa) &\leq \left(\frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(\varepsilon_1) + \left(\frac{\kappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(\varepsilon_2) \\ &\quad - \left(\frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\kappa - \varepsilon_1|) - \left(\frac{\kappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\varepsilon_2 - \kappa|).\end{aligned}$$

Replacing κ by $\varepsilon_1 + \varepsilon_2 - \kappa$ in the above inequality and adding the two resulting inequalities completes the proof. $\square$

On the basis of aforementioned Lemmas, we prove the inequalities of Fejér-type for s -superquadratic function.

Theorem 2.5. *Let $P \in L^2(I)$ be nonnegative and symmetric about $\frac{\varepsilon_1 + \varepsilon_2}{2}$. Suppose that $\Phi : I \rightarrow (0, \infty)$ is an s -superquadratic integrable stochastic function. Then, for every $\varepsilon_1, \varepsilon_2 \in I$ and $s \in (0, 1]$,*

$$\begin{aligned}&2^{s-1}\Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2} P(\kappa) d\kappa + \int_{\varepsilon_1}^{\varepsilon_2} \Phi\left(\left|\kappa - \frac{\varepsilon_1 + \varepsilon_2}{2}\right|\right) P(\kappa) d\kappa \\ &\leq \int_{\varepsilon_1}^{\varepsilon_2} \Phi(\kappa) P(\kappa) d\kappa \\ &\leq \left(\Phi(\varepsilon_1, \cdot) + \Phi(\varepsilon_2, \cdot)\right) \int_{\varepsilon_1}^{\frac{\varepsilon_1 + \varepsilon_2}{2}} \left[\left(\frac{\kappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s + \left(\frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1}\right)^s\right] P(\kappa) d\kappa \\ &\quad - \int_{\varepsilon_1}^{\varepsilon_2} \left[\left(\frac{\kappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\varepsilon_2 - \kappa|) + \left(\frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\kappa - \varepsilon_1|)\right] P(\kappa) d\kappa.\end{aligned}$$

Proof. Applying Lemma 2.3 gives

$$2^s \Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \leq \Phi(\kappa) + \Phi(\varepsilon_1 + \varepsilon_2 - \kappa) - 2\Phi\left(\left|\kappa - \frac{\varepsilon_1 + \varepsilon_2}{2}\right|\right).$$

Multiplying by $P(\kappa)$ and integrating over $[\varepsilon_1, \varepsilon_2]$, the symmetry $P(\kappa) = P(\varepsilon_1 + \varepsilon_2 - \kappa)$ yields

$$\begin{aligned}&2^{s-1}\Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2} P(\kappa) d\kappa + \int_{\varepsilon_1}^{\varepsilon_2} \Phi\left(\left|\kappa - \frac{\varepsilon_1 + \varepsilon_2}{2}\right|\right) P(\kappa) d\kappa \\ &\leq \int_{\varepsilon_1}^{\varepsilon_2} \Phi(\kappa) P(\kappa) d\kappa.\end{aligned}\tag{6}$$

For the upper bound, Lemma 2.4 implies

$$\Phi(\kappa) + \Phi(\varepsilon_1 + \varepsilon_2 - \kappa) \leq \mathcal{A}(\kappa) - 2\mathcal{B}(\kappa),$$

where

$$\mathcal{A}(\kappa) = (\Phi(\varepsilon_1) + \Phi(\varepsilon_2)) \left[\left(\frac{\kappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s + \left(\frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1}\right)^s \right],$$

and

$$\mathcal{B}(\kappa) = \left(\frac{\kappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\varepsilon_2 - \kappa|) + \left(\frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1}\right)^s \Phi(|\kappa - \varepsilon_1|).$$

Integrating both sides against $P(\kappa)$ and using the symmetry of P gives

$$2 \int_{\varepsilon_1}^{\varepsilon_2} \Phi(\kappa) P(\kappa) d\kappa \leq 2(\Phi(\varepsilon_1, \cdot) + \Phi(\varepsilon_2)) \int_{\varepsilon_1}^{\frac{\varepsilon_1 + \varepsilon_2}{2}} \left[\left(\frac{\kappa - \varepsilon_1}{\varepsilon_2 - \varepsilon_1} \right)^s + \left(\frac{\varepsilon_2 - \kappa}{\varepsilon_2 - \varepsilon_1} \right)^s \right] P(\kappa) d\kappa - 2 \int_{\varepsilon_1}^{\varepsilon_2} \mathcal{B}(\kappa) P(\kappa) d\kappa. \quad (7)$$

Dividing by 2 establishes the upper estimate. Combining it with the lower estimate completes the proof. $\square$

3. Application to Fatigue Mechanics

Fatigue is one of the most common failure mechanisms in engineering structures subjected to repeated or cyclic loading. During cyclic deformation, microscopic damage gradually accumulates until a critical level is reached, leading to crack initiation and eventual structural failure. Since fatigue damage is cumulative in nature, it can naturally be modeled by weighted integral functionals. In this regard, the established Fejér-type inequality for s -superquadratic functions provides computable upper and lower bounds for the accumulated fatigue damage.

Let $\kappa \in [\varepsilon_1, \varepsilon_2]$ denote the loading cycle (or a normalized cycle parameter). Suppose that $\Phi(\kappa)$ represents the fatigue damage density generated during the loading cycle κ . The total accumulated fatigue damage over the interval $[\varepsilon_1, \varepsilon_2]$ is defined by

$$D = \int_{\varepsilon_1}^{\varepsilon_2} \Phi(\kappa) P(\kappa) d\kappa, \quad (8)$$

where $P(\kappa) \geq 0$ is an integrable weighting function representing the probability of load occurrence, material reliability, or a cycle-weighting factor. Observe that the quantity in (8) coincides exactly with the middle integral appearing in Theorem 2.5.

Experimental investigations indicate that fatigue damage generally evolves in a nonlinear manner as the number of loading cycles increases. For illustrative purposes, consider the fatigue damage density $\Phi(\kappa) = \kappa^3$.

Since cubic functions are superquadratic, they satisfy the framework of s -superquadraticity for an appropriate parameter $s = \frac{1}{2}$. Assume $\varepsilon_1 = 1$, and $\varepsilon_2 = 2$. The midpoint of the interval is $\frac{\varepsilon_1 + \varepsilon_2}{2} = \frac{3}{2}$.

Choose the symmetric weighting function

$$P(\kappa) = \left(\kappa - \frac{3}{2} \right)^2 + 1.$$

Then

$$P\left(\frac{3}{2} + x\right) = P\left(\frac{3}{2} - x\right),$$

which verifies the symmetry assumption required in Theorem 2.5. Expanding,

$$P(\kappa) = \kappa^2 - 3\kappa + \frac{13}{4}.$$

Using the above expression,

$$\int_1^2 P(\kappa) d\kappa = \int_1^2 \left(\kappa^2 - 3\kappa + \frac{13}{4} \right) d\kappa = \left[\frac{\kappa^3}{3} - \frac{3}{2}\kappa^2 + \frac{13}{4}\kappa \right]_1^2 = \frac{13}{12}.$$

First, we compute the lower estimate. Since $\Phi\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$, and $2^{s-1} = 2^{-1/2} = \frac{1}{\sqrt{2}}$, we obtain

$$L_1 = 2^{-1/2} \Phi\left(\frac{3}{2}\right) \int_1^2 P(\kappa) d\kappa = 2^{-1/2} \cdot \frac{27}{8} \cdot \frac{13}{12} = \frac{351}{96\sqrt{2}} \approx 2.585.$$

The second integral appearing in the lower bound is $L_2 = \int_1^2 \left|\kappa - \frac{3}{2}\right|^3 P(\kappa) d\kappa$. Introduce the substitution $u = \kappa - \frac{3}{2}$. Then $u \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, and consequently

$$\begin{aligned} L_2 &= 2 \int_0^{1/2} u^3(u^2 + 1) du = 2 \int_0^{1/2} (u^5 + u^3) du \\ &= 2 \left[\frac{u^6}{6} + \frac{u^4}{4} \right]_0^{1/2} = 2 \left(\frac{1}{384} + \frac{1}{64} \right) = \frac{7}{192} \approx 0.03646. \end{aligned}$$

Lower Fatigue-Damage Bound $L = L_1 + L_2 = 2.62146$.

Now, we compute the exact fatigue damage:

$$\begin{aligned} D &= \int_1^2 \kappa^3 \left(\kappa^2 - 3\kappa + \frac{13}{4} \right) d\kappa = \int_1^2 \left(\kappa^5 - 3\kappa^4 + \frac{13}{4}\kappa^3 \right) d\kappa \\ &= \left[\frac{\kappa^6}{6} - \frac{3}{5}\kappa^5 + \frac{13}{16}\kappa^4 \right]_1^2 = 4.02917. \end{aligned}$$

Hence, $D=4.02917$.

Next, we perform the computation of the upper bound of the Theorem 2.5.

Since $\left(\frac{\kappa-\varepsilon_1}{\varepsilon_2-\varepsilon_1}\right)^s = (\kappa-1)^{1/2}$, and $\left(\frac{\varepsilon_2-\kappa}{\varepsilon_2-\varepsilon_1}\right)^s = (2-\kappa)^{1/2}$.

Moreover, $\Phi(1) = 1$, and $\Phi(2) = 8$, which implies that $\Phi(1) + \Phi(2) = 9$. Substituting these values into the right-hand side of Theorem 2.5, we obtain $U = 9I_1 - I_2$, where

$$I_1 = \int_1^{\frac{3}{2}} P(\kappa) [(\kappa-1)^{1/2} + (2-\kappa)^{1/2}] d\kappa = 1.128273,$$

and

$$I_2 = \int_1^2 [(\kappa-1)^{1/2}(2-\kappa)^3 + (2-\kappa)^{1/2}(\kappa-1)^3] P(\kappa) d\kappa = 3.420336.$$

Therefore,

$$U = 9I_1 - I_2 = 9(1.128273) - 3.420336 = 10.154457 - 3.420336 = 6.734121.$$

Consequently,

$$2.62146 \leq 4.02917 \leq 6.734121.$$

This confirms that the exact fatigue damage lies within the theoretical bounds predicted by the proposed Fejér-type inequality. The established Fejér-type inequality possesses a clear engineering significance in fatigue mechanics.

- The middle integral represents the exact accumulated fatigue damage over the prescribed loading interval.
- The left-hand side provides a guaranteed lower estimate of the accumulated damage by utilizing the midpoint fatigue damage together with the local deviation from the midpoint.

- The right-hand side yields a computable upper estimate depending only on the endpoint fatigue damages and the correction terms arising from the s -superquadratic property.
- Consequently, the proposed inequality enables engineers to estimate the cumulative fatigue damage without evaluating the damage density at every loading cycle, thereby reducing computational cost while providing mathematically rigorous lower and upper bounds. Such estimates are particularly valuable in fatigue-life prediction, structural reliability assessment, and maintenance scheduling of engineering components subjected to cyclic loading.

4. Conclusions

In this paper, the new class of s -superquadratic functions was employed as the underlying framework for establishing several new Fejér-type integral inequalities. By utilizing two fundamental lemmas associated with this class, refined weighted integral inequalities were derived. An application to fatigue mechanics was also presented, demonstrating the effectiveness of the obtained inequalities in providing rigorous lower and upper bounds for cumulative fatigue damage under cyclic loading. The proposed results furnish useful analytical tools for the qualitative and quantitative analysis of nonlinear integral inequalities and their engineering applications.

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