



Research Article

Symmetric Post-Quantum Ostrowski-Type Inequalities

Muhammad Nasim Aftab¹, Christophe Chesneau², Sheza El-Deeb^{3,4}, Silvestru Sever Dragomir⁵ and Mehmet Zeki Sarikaya⁶

¹Department of Mathematics, University of Engineering and Technology, Lahore 54890, Pakistan

²Department of Mathematics, LMNO, University of Caen Normandie, Caen 14032, Caen, France

³Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia

⁴Department of Mathematics, Faculty of Science, Damietta University, New Damietta, Egypt

⁵Applied Mathematics Research Group, ISILC, Victoria University, PO Box 14428, Melbourne City, MC 8001, VIC, Australia

⁶Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Türkiye

* Corresponding author: mzekisarikaya@duzce.edu.tr

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Abstract

In this manuscript, we establish a Montgomery-type integral identity in the framework of left symmetric post-quantum calculus. Using this identity, we derive Ostrowski-type inequalities under suitable convexity assumptions on the symmetric post-quantum derivative. In addition, a Hölder-type estimate is obtained for conjugate exponents. The limiting case $p \rightarrow 1^-$ is also discussed, recovering the corresponding symmetric quantum inequalities.

1. Introduction

The theory of convex functions has been investigated in significant detail over the last several years because of its relevance to many areas of the pure and applied sciences. Owing to the strong relationship between the theory of inequalities and the theory of convex functions, many inequalities that are proven for convex and differentiable convex functions can be explored in the published material; see [1, 2, 3]. Numerous inequalities for various convex functions have been established and can be found in [4, 5, 6, 7]. In the following, the Ostrowski-type inequalities for functions whose corresponding absolute value derivatives are s -convex have been established in [8], while certain assumptions and applications in numerical integration of the Ostrowski-type inequalities are studied in [9] and [10]. Moreover, Ostrowski-type inequalities for generalized exponentially convex functions [11], hypergeometric functions [12], and n -polynomial exponentially s -convex functions [13] are also studied. The classical Ostrowski inequality can be expressed as.

Theorem 1.1. [8] Let $\mathcal{L} : \mathbb{I} \subset [0, \infty[\rightarrow \mathbb{R}$ be the differentiable function on the interior of $\mathbb{I}$ and $\mathcal{L}' \in L[\kappa, \theta]$. Assume that $|\mathcal{L}'(\gamma)| \leq \mathcal{M}$, then for all $\gamma \in [\kappa, \theta]$

$$\left| \mathcal{L}(\gamma) - \frac{\int_{\kappa}^{\theta} \mathcal{L}(v) dv}{\theta - \kappa} \right| \leq \frac{\{(\gamma - \kappa)^2 + (\theta - \gamma)^2\} \mathcal{M}}{2(\theta - \kappa)} \quad (1)$$

holds.

The concepts of the quantum difference operator and its inverse on finite intervals were first presented in [14, 15]. Numerous quantum versions of classical integral inequalities, including Hölder's, Ostrowski's, and Hermite-Hadamard's inequalities, were created by Tariboon et al. [14]. Meanwhile, the new quantum variants of Ostrowski's inequality for q -differentiable convex functions were established by Noor et al. [16]. In addition, its symmetrized quantum analogue is studied in [17, 18]. Furthermore, the post-quantum analogue of Ostrowski's inequality for twice (p, q) -differentiable convex functions is constructed by Luangboon et al. [19]. For further knowledge on the hierarchy of quantum calculus and its symmetric variants with their applications to statistical and inequality theory, we recommend [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. The main contribution of this paper is the establishment of a Montgomery-type identity for the left (p, q) -symmetric integral. This identity is then used to derive two Ostrowski-type estimates, including a Hölder-type version under convexity assumptions. The limiting case $p \rightarrow 1^-$ connects the results with the corresponding symmetric quantum inequalities.

2. Material and Methods

Throughout this paper, unless otherwise stated, we assume that the parameters p and q satisfy $0 < q < p \leq 1$. All definitions and results below are considered under these assumptions. The symmetrized model of quantum calculus is called symmetric quantum or q -symmetric calculus [20]. The left-point extension of any real interval $[\kappa, \theta]$ for a symmetric quantum calculus is known as ${}_{\kappa}q$ -symmetric, or a left q -symmetric calculus, and several left-symmetric quantum integral results for convex and coordinated convex functions are investigated in [17, 27, 28]. Some of them are stated below.

Lemma 2.1. [17] Let $\mathcal{L} : \mathcal{I} \subset \mathbb{R} \longrightarrow \mathbb{R}$ be the left q -symmetric differentiable function on the interior of $\mathcal{I}$ with $\kappa, \theta \in \mathcal{I}$ such that $\kappa < \theta$. Assume that ${}_{\kappa}\widetilde{\mathcal{D}}_q\mathcal{L} \in L[\kappa, \theta]$, then

$$\frac{\mathcal{L}(\gamma) - \frac{q \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\widetilde{\mathcal{D}}_q v}{\theta - \kappa}}{q} = \frac{1}{\theta - \kappa} \left[(\gamma - \kappa)^2 \int_0^1 v {}_{\kappa}\widetilde{\mathcal{D}}_q \mathcal{L}(\gamma v + (1 - v)\kappa) {}_0\widetilde{\mathcal{D}}_q v \right. \\ \left. + (\theta - \gamma)^2 \int_0^1 v {}_{\kappa}\widetilde{\mathcal{D}}_q \mathcal{L}(\gamma v + (1 - v)\theta) {}_0\widetilde{\mathcal{D}}_q v \right] \quad (2)$$

equality holds.

Theorem 2.2. [17] Let $\mathcal{L} : \mathcal{I} \longrightarrow \mathbb{R}$ be the left q -symmetric differentiable function on (κ, θ) and ${}_{\kappa}\widetilde{\mathcal{D}}_q\mathcal{L} \in L[\kappa, \theta]$. For $r_1, r_2 > 1$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$, assume that $|{}_{\kappa}\widetilde{\mathcal{D}}_q\mathcal{L}(\gamma)|^{r_1}$ is a convex function on $[\kappa, \theta]$ and $|{}_{\kappa}\widetilde{\mathcal{D}}_q\mathcal{L}(\gamma)| \leq \mathcal{M}$, then for all $\gamma \in [\kappa, \theta]$

$$\left| \frac{\mathcal{L}(\gamma) - \frac{q \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\widetilde{\mathcal{D}}_q v}{\theta - \kappa}}{q} \right| \leq \frac{\{(\gamma - \kappa)^2 + (\theta - \gamma)^2\} \mathcal{M}}{(\theta - \kappa) [\widetilde{r_2 + 1}]_q^{\frac{1}{r_2}}} \quad (3)$$

holds.

In 2025, Aftab et al. [33] defined the symmetric post-quantum number and differential operator, which are mentioned below.

Definition 2.3. [33] If r is a non-negative integer, then the symmetric post-quantum number can be defined as

$$[\widetilde{r}]_{p,q} = \frac{(pq)^r - (pq)^{-r}}{pq - (pq)^{-1}}.$$

If $p \longrightarrow 1^-$, then it becomes a symmetric quantum number.

Although Definition 2.3 is initially stated for non-negative integers, in the sequel the same closed-form expression is used for real arguments whenever it is well-defined. In particular, this convention is used for $r_2 + 1$ in Theorem 3.5.

Definition 2.4. [33] Let $\mathcal{L}(\gamma)$ be any function; then the symmetric post-quantum differential can be defined as

$$\widetilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma) = \mathcal{L}(pq\gamma) - \mathcal{L}(p^{-1}q^{-1}\gamma).$$

If $p \longrightarrow 1^-$, then it becomes a symmetric quantum differential.

These definitions have become the foundation for developing the symmetric post-quantum or (p, q) -symmetric calculus. In the following, ${}_{\kappa}(p, q)$ -symmetric, or a left (p, q) -symmetric calculus, is the left-point extension of any real interval $[\kappa, \theta]$ for a symmetric post-quantum calculus. Its various left symmetric post-quantum integral inequalities are examined in [34, 35]. The definitions of left symmetric post-quantum derivative and definite integrals are stated below.

Definition 2.5. [34] Let $\mathcal{L}$ be a continuous function, then its left symmetric post-quantum derivative at $\gamma \in [\kappa, \theta]$ is expressed as

$${}_{\kappa}\widetilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma) = \frac{\mathcal{L}((1 - pq)\kappa + pq\gamma) - \mathcal{L}((1 - p^{-1}q^{-1})\kappa + p^{-1}q^{-1}\gamma)}{(pq - p^{-1}q^{-1})(\gamma - \kappa)}, \quad \gamma \neq \kappa. \quad (4)$$

Definition 2.6. [34] Let $\mathcal{L}$ be a continuous function, then its left symmetric post-quantum definite integral on $[\kappa, \theta]$ can be defined as

$$\int_{\kappa}^{\gamma} \mathcal{L}(v) {}_{\kappa}\widetilde{\mathfrak{d}}_{p,q} v = (1 - p^2 q^2)(\gamma - \kappa) \sum_{r=0}^{\infty} p^{2r} q^{2r} \mathcal{L}((1 - p^{2r+1} q^{2r+1})\kappa + p^{2r+1} q^{2r+1} \gamma), \quad (5)$$

for $\gamma \in [\kappa, \theta]$.

The corresponding Hölder inequality can be formulated as follows.

Theorem 2.7. [34] Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the left symmetric post-quantum integrable functions on $[\kappa, \theta]$, then

$$\int_{\kappa}^{\theta} |\mathcal{L}_1(v) \mathcal{L}_2(v)| {}_{\kappa}\widetilde{\mathfrak{d}}_{p,q} v \leq \left(\int_{\kappa}^{\theta} |\mathcal{L}_1(v)|^{r_1} {}_{\kappa}\widetilde{\mathfrak{d}}_{p,q} v \right)^{\frac{1}{r_1}} \left(\int_{\kappa}^{\theta} |\mathcal{L}_2(v)|^{r_2} {}_{\kappa}\widetilde{\mathfrak{d}}_{p,q} v \right)^{\frac{1}{r_2}} \quad (6)$$

exists for $r_1, r_2 > 1$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$.

3. Main Results

First, we establish the Montgomery identity in the left (p, q) -symmetric calculus.

Lemma 3.1. Let $\mathcal{L} : \mathbb{I} \subset \mathfrak{R} \longrightarrow \mathfrak{R}$ be the left (p, q) -symmetric differentiable function on the interior of $\mathbb{I}$ with $\kappa, \theta \in \mathbb{I}$ such that $\kappa < \theta$. Assume that ${}_{\kappa}\widetilde{\mathcal{D}}_{p,q} \mathcal{L} \in L[\kappa, \theta]$, then

$$\begin{aligned} \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\widetilde{\mathfrak{d}}_{p,q} v}{\theta - \kappa}}{pq} &= \frac{1}{\theta - \kappa} \left[(\gamma - \kappa)^2 \int_0^1 v {}_{\kappa}\widetilde{\mathcal{D}}_{p,q} \mathcal{L}(\gamma v + (1 - v)\kappa) {}_0\widetilde{\mathfrak{d}}_{p,q} v \right. \\ &\quad \left. + (\theta - \gamma)^2 \int_0^1 v {}_{\kappa}\widetilde{\mathcal{D}}_{p,q} \mathcal{L}(\gamma v + (1 - v)\theta) {}_0\widetilde{\mathfrak{d}}_{p,q} v \right] \end{aligned} \quad (7)$$

equality holds.

Proof. Suppose that $\mathbb{I}_1 = \frac{(\gamma - \kappa)^2}{\theta - \kappa} \int_0^1 v {}_{\kappa}\widetilde{\mathcal{D}}_{p,q} \mathcal{L}(\gamma v + (1 - v)\kappa) {}_0\widetilde{\mathfrak{d}}_{p,q} v$ and

$\mathbb{I}_2 = \frac{(\theta - \gamma)^2}{\theta - \kappa} \int_0^1 v {}_{\kappa}\widetilde{\mathcal{D}}_{p,q} \mathcal{L}(\gamma v + (1 - v)\theta) {}_0\widetilde{\mathfrak{d}}_{p,q} v$, then using (4) and (5)

$$\begin{aligned} \mathbb{I}_1 &= \frac{(\gamma - \kappa)^2}{\theta - \kappa} \int_0^1 v {}_{\kappa}\widetilde{\mathcal{D}}_{p,q} \mathcal{L}(\gamma v + (1 - v)\kappa) {}_0\widetilde{\mathfrak{d}}_{p,q} v \\ &= \frac{(\gamma - \kappa)^2}{\theta - \kappa} \int_0^1 v \frac{\mathcal{L}(\gamma v p q + (1 - v p q)\kappa) - \mathcal{L}(\gamma v p^{-1} q^{-1} + (1 - v p^{-1} q^{-1})\kappa)}{(p q - p^{-1} q^{-1})(\gamma - \kappa)v} {}_0\widetilde{\mathfrak{d}}_{p,q} v \\ &= \frac{(\gamma - \kappa)}{(\theta - \kappa)(p q - p^{-1} q^{-1})} \\ &\quad \times \int_0^1 \mathcal{L}(\gamma v p q + (1 - v p q)\kappa) - \mathcal{L}(\gamma v p^{-1} q^{-1} + (1 - v p^{-1} q^{-1})\kappa) {}_0\widetilde{\mathfrak{d}}_{p,q} v \end{aligned}$$

$$\begin{aligned}
 &= \frac{(\gamma - \kappa)}{(\theta - \kappa)(pq - p^{-1}q^{-1})} \\
 &\times \left[(p^{-1}q^{-1} - pq) \sum_{r=0}^{\infty} p^{2r+1} q^{2r+1} \mathcal{L}(\gamma p^{2r+2} q^{2r+2} + (1 - p^{2r+2} q^{2r+2})\kappa) \right. \\
 &\quad \left. - (p^{-1}q^{-1} - pq) \sum_{r=0}^{\infty} p^{2r+1} q^{2r+1} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) \right] \\
 &= \frac{\gamma - \kappa}{\theta - \kappa} \left[-p^{-1}q^{-1} \sum_{r=0}^{\infty} p^{2r+2} q^{2r+2} \mathcal{L}(\gamma p^{2r+2} q^{2r+2} + (1 - p^{2r+2} q^{2r+2})\kappa) \right. \\
 &\quad \left. + pq \sum_{r=0}^{\infty} p^{2r} q^{2r} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) \right].
 \end{aligned}$$

Changing the summation index from $r + 1$ to r , we obtain

$$\begin{aligned}
 &= \frac{\gamma - \kappa}{\theta - \kappa} \left[-p^{-1}q^{-1} \sum_{r=1}^{\infty} p^{2r} q^{2r} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) \right. \\
 &\quad \left. + pq \sum_{r=0}^{\infty} p^{2r} q^{2r} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) \right] \\
 &= \frac{\gamma - \kappa}{\theta - \kappa} \left[-p^{-1}q^{-1} \sum_{r=0}^{\infty} p^{2r} q^{2r} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) + p^{-1}q^{-1} \mathcal{L}(\gamma) \right. \\
 &\quad \left. + pq \sum_{r=0}^{\infty} p^{2r} q^{2r} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) \right].
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{I}_1 &= \frac{\gamma - \kappa}{\theta - \kappa} \left[-p^{-1}q^{-1} \sum_{r=0}^{\infty} p^{2r+1} q^{2r+1} \frac{1}{pq} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) + p^{-1}q^{-1} \mathcal{L}(\gamma) \right. \\
 &\quad \left. + pq \sum_{r=0}^{\infty} p^{2r+1} q^{2r+1} \frac{1}{pq} \mathcal{L}(\gamma p^{2r} q^{2r} + (1 - p^{2r} q^{2r})\kappa) \right] \\
 &= \frac{\gamma - \kappa}{\theta - \kappa} \left[-p^{-1}q^{-1} \sum_{r=0}^{\infty} p^{2r+1} q^{2r+1} \mathcal{L}(\gamma p^{2r+1} q^{2r+1} + (1 - p^{2r+1} q^{2r+1})\kappa) + p^{-1}q^{-1} \mathcal{L}(\gamma) \right. \\
 &\quad \left. + pq \sum_{r=0}^{\infty} p^{2r+1} q^{2r+1} \mathcal{L}(\gamma p^{2r+1} q^{2r+1} + (1 - p^{2r+1} q^{2r+1})\kappa) \right] \\
 &= \frac{\gamma - \kappa}{\theta - \kappa} \left[-(p^{-1}q^{-1} - pq) \sum_{r=0}^{\infty} p^{2r+1} q^{2r+1} \mathcal{L}(\gamma p^{2r+1} q^{2r+1} + (1 - p^{2r+1} q^{2r+1})\kappa) \right. \\
 &\quad \left. + p^{-1}q^{-1} \mathcal{L}(\gamma) \right] \\
 &= \frac{\gamma - \kappa}{\theta - \kappa} \left[\frac{\mathcal{L}(\gamma)}{pq} - \frac{1}{\gamma - \kappa} \int_{\kappa}^{\gamma} \mathcal{L}(\mathbf{v}) {}_{\kappa}\widetilde{\mathbf{d}}_{p,q} \mathbf{v} \right] \\
 &= \frac{(\gamma - \kappa) \mathcal{L}(\gamma)}{(\theta - \kappa) pq} - \frac{1}{\theta - \kappa} \int_{\kappa}^{\gamma} \mathcal{L}(\mathbf{v}) {}_{\kappa}\widetilde{\mathbf{d}}_{p,q} \mathbf{v}.
 \end{aligned} \tag{8}$$

Similarly,

$$I_2 = \frac{(\theta - \gamma)\mathcal{L}(\gamma)}{(\theta - \kappa)pq} - \frac{1}{\theta - \kappa} \int_{\gamma}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q}v. \quad (9)$$

Adding (8) and (9)

$$\begin{aligned} I_1 + I_2 &= \frac{(\gamma - \kappa)\mathcal{L}(\gamma)}{(\theta - \kappa)pq} - \frac{1}{\theta - \kappa} \int_{\kappa}^{\gamma} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q}v + \frac{(\theta - \gamma)\mathcal{L}(\gamma)}{(\theta - \kappa)pq} - \frac{1}{\theta - \kappa} \int_{\gamma}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q}v \\ &= \frac{\mathcal{L}(\gamma)}{pq} - \frac{1}{\theta - \kappa} \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q}v. \end{aligned}$$

Hence (7) is achieved. $\square$

Remark 3.2. If $p \rightarrow 1^-$ in Lemma 3.1, then (2) is obtained.

The following Ostrowski-type inequalities can now be constructed using Lemma 3.1.

Theorem 3.3. Let $\mathcal{L} : \mathbb{I} \rightarrow \mathbb{R}$ be the left (p, q) -symmetric differentiable function on (κ, θ) and ${}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L} \in L[\kappa, \theta]$. Assume that $|{}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma)|$ is convex function on $[\kappa, \theta]$ and $|{}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma)| \leq \mathcal{M}$, then for all $\gamma \in [\kappa, \theta]$

$$\left| \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q}v}{\theta - \kappa}}{pq} \right| \leq \frac{pq\{(\gamma - \kappa)^2 + (\theta - \gamma)^2\}\mathcal{M}}{(\theta - \kappa)(1 + p^2q^2)} \quad (10)$$

holds.

Proof. Applying the modulus on (7)

$$\begin{aligned} \left| \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q}v}{\theta - \kappa}}{pq} \right| &\leq \frac{(\gamma - \kappa)^2}{\theta - \kappa} \left| \int_0^1 v {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\kappa) {}_0\tilde{\mathfrak{d}}_{p,q}v \right| \\ &\quad + \frac{(\theta - \gamma)^2}{\theta - \kappa} \left| \int_0^1 v {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\theta) {}_0\tilde{\mathfrak{d}}_{p,q}v \right| \\ &\leq \frac{(\gamma - \kappa)^2}{\theta - \kappa} \int_0^1 \left| v {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\kappa) \right| {}_0\tilde{\mathfrak{d}}_{p,q}v \\ &\quad + \frac{(\theta - \gamma)^2}{\theta - \kappa} \int_0^1 \left| v {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\theta) \right| {}_0\tilde{\mathfrak{d}}_{p,q}v. \end{aligned}$$

Using the convexity of $|{}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma)|$

$$\begin{aligned} &\left| \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q}v}{\theta - \kappa}}{pq} \right| \\ &\leq \frac{(\gamma - \kappa)^2}{\theta - \kappa} \int_0^1 v^2 \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma) \right| {}_0\tilde{\mathfrak{d}}_{p,q}v + \int_0^1 v(1 - v) \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\kappa) \right| {}_0\tilde{\mathfrak{d}}_{p,q}v \\ &\quad + \frac{(\theta - \gamma)^2}{\theta - \kappa} \int_0^1 v^2 \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma) \right| {}_0\tilde{\mathfrak{d}}_{p,q}v + \int_0^1 v(1 - v) \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\theta) \right| {}_0\tilde{\mathfrak{d}}_{p,q}v \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\mathcal{M}(\gamma - \kappa)^2}{\theta - \kappa} \int_0^1 v {}_0\tilde{\mathfrak{d}}_{p,q} v + \frac{\mathcal{M}(\theta - \gamma)^2}{\theta - \kappa} \int_0^1 v {}_0\tilde{\mathfrak{d}}_{p,q} v \\
 &= \frac{\mathcal{M}\{(\gamma - \kappa)^2 + (\theta - \gamma)^2\}}{\theta - \kappa} \int_0^1 v {}_0\tilde{\mathfrak{d}}_{p,q} v \\
 &= \frac{pq\{(\gamma - \kappa)^2 + (\theta - \gamma)^2\}\mathcal{M}}{(\theta - \kappa)(1 + p^2 q^2)}.
 \end{aligned}$$

□

Remark 3.4. If $p \rightarrow 1^-$ in Theorem 3.3, it becomes the Ostrowski inequality for the symmetric quantum calculus.

Theorem 3.5. Let $\mathcal{L} : \mathbb{I} \rightarrow \mathbb{R}$ be the left (p, q) -symmetric differentiable function on (κ, θ) and ${}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L} \in L[\kappa, \theta]$. For $r_1, r_2 > 1$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$, assume that $|{}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma)|^{r_1}$ is a convex function on $[\kappa, \theta]$ and $|{}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma)| \leq \mathcal{M}$, then for all $\gamma \in [\kappa, \theta]$

$$\left| \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q} v}{\theta - \kappa}}{pq} \right| \leq \frac{\{(\gamma - \kappa)^2 + (\theta - \gamma)^2\}\mathcal{M}}{(\theta - \kappa) \widetilde{[r_2 + 1]_{p,q}^{\frac{1}{r_2}}}} \quad (11)$$

holds.

Proof. Applying the modulus on (7)

$$\begin{aligned}
 \left| \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q} v}{\theta - \kappa}}{pq} \right| &\leq \frac{(\gamma - \kappa)^2}{\theta - \kappa} \int_0^1 \left| v {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\kappa) \right| {}_0\tilde{\mathfrak{d}}_{p,q} v \\
 &\quad + \frac{(\theta - \gamma)^2}{\theta - \kappa} \int_0^1 \left| v {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\theta) \right| {}_0\tilde{\mathfrak{d}}_{p,q} v.
 \end{aligned}$$

Applying (6) on the right-hand side

$$\begin{aligned}
 &\left| \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q} v}{\theta - \kappa}}{pq} \right| \\
 &\leq \frac{(\gamma - \kappa)^2}{\theta - \kappa} \left(\int_0^1 v^{r_2} {}_0\tilde{\mathfrak{d}}_{p,q} v \right)^{\frac{1}{r_2}} \left(\int_0^1 \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\kappa) \right|^{r_1} {}_0\tilde{\mathfrak{d}}_{p,q} v \right)^{\frac{1}{r_1}} \\
 &\quad + \frac{(\theta - \gamma)^2}{\theta - \kappa} \left(\int_0^1 v^{r_2} {}_0\tilde{\mathfrak{d}}_{p,q} v \right)^{\frac{1}{r_2}} \left(\int_0^1 \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma v + (1 - v)\theta) \right|^{r_1} {}_0\tilde{\mathfrak{d}}_{p,q} v \right)^{\frac{1}{r_1}}.
 \end{aligned}$$

Using the convexity of $|{}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma)|^{r_1}$

$$\begin{aligned}
 \left| \frac{\mathcal{L}(\gamma) - \frac{pq \int_{\kappa}^{\theta} \mathcal{L}(v) {}_{\kappa}\tilde{\mathfrak{d}}_{p,q} v}{\theta - \kappa}}{pq} \right| &\leq \frac{(\gamma - \kappa)^2}{\theta - \kappa} \left(\frac{1}{\widetilde{[r_2 + 1]_{p,q}}} \right)^{\frac{1}{r_2}} \\
 &\quad \times \left(\int_0^1 v \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\gamma) \right|^{r_1} + (1 - v) \left| {}_{\kappa}\tilde{\mathcal{D}}_{p,q}\mathcal{L}(\kappa) \right|^{r_1} {}_0\tilde{\mathfrak{d}}_{p,q} v \right)^{\frac{1}{r_1}} \\
 &\quad + \frac{(\theta - \gamma)^2}{\theta - \kappa} \left(\frac{1}{\widetilde{[r_2 + 1]_{p,q}}} \right)^{\frac{1}{r_2}}
 \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^1 v \left| {}_{\kappa} \widetilde{\mathcal{D}}_{p,q} \mathcal{L}(\gamma) \right|^{\mathbf{r}_1} + (1-v) \left| {}_{\kappa} \widetilde{\mathcal{D}}_{p,q} \mathcal{L}(\theta) \right|^{\mathbf{r}_1} {}_0 \widetilde{\mathcal{D}}_{p,q} v \right)^{\frac{1}{\mathbf{r}_1}} \\
& \leq \frac{(\gamma - \kappa)^2}{\theta - \kappa} \left(\frac{1}{[\mathbf{r}_2 + 1]_{p,q}} \right)^{\frac{1}{\mathbf{r}_2}} (\mathcal{M}^{\mathbf{r}_1})^{\frac{1}{\mathbf{r}_1}} \left(\int_0^1 1 {}_0 \widetilde{\mathcal{D}}_{p,q} v \right)^{\frac{1}{\mathbf{r}_1}} \\
& + \frac{(\theta - \gamma)^2}{\theta - \kappa} \left(\frac{1}{[\mathbf{r}_2 + 1]_{p,q}} \right)^{\frac{1}{\mathbf{r}_2}} (\mathcal{M}^{\mathbf{r}_1})^{\frac{1}{\mathbf{r}_1}} \left(\int_0^1 1 {}_0 \widetilde{\mathcal{D}}_{p,q} v \right)^{\frac{1}{\mathbf{r}_1}} \\
& = \frac{\{(\gamma - \kappa)^2 + (\theta - \gamma)^2\} \mathcal{M}}{(\theta - \kappa) [\mathbf{r}_2 + 1]_{p,q}^{\frac{1}{\mathbf{r}_2}}}.
\end{aligned}$$

□

Remark 3.6. If $p \rightarrow 1^-$ in (11) of Theorem 3.5, then it reduces to (3) of Theorem 2.2.

4. Conclusion

In this research, we constructed the symmetric post-quantum Montgomery integral identity. Following suitable convexity assumptions on the symmetric post-quantum derivative, we derive Ostrowski-type inequalities using this identity. Furthermore, an estimate of the Hölder-type is obtained for conjugate exponents. Moreover, the obtained results reduce to their corresponding symmetric quantum versions as $p \rightarrow 1^-$. Future studies in the fields of integral inequality, symmetric quantum calculus, post-quantum calculus, symmetric post-quantum calculus, and their generalized perspectives could expand on the foundation of this framework.

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