Int. J. Appl. Math. Optim. AI, 1(1), (2026), 40-45 https://doi.org/10.XXXXX/ijamoai.xxxxxx
ISSN: XXXX-XXXX https://www.ijamoai.org

International Journal of Applied Mathematics, Optimization and AI

Research Article

Convergence Analysis of HA-Iteration for Enriched Nonexpan- -sive Mappings in Banach Space Settings

Samra Asjad[1], **Hanif Ullah**[2], **Mubashar Arshad**[3,4,*], **Evren Hincal** [5] **and Ikram Ullah**[6]

[1]*Department of Mathematics, The University of Alabama at Birmingham, Birmingham, AL, USA.*
[2]*Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat, KPK, Pakistan*
[3]*Institute for Numerical and Applied Mathematics, University of Göttingen, Göttingen, 37083, Germany*
[4]*Department of mathematics, University of Genoa Genoa, Italy*
[5]*Mathematics Research Center, Near East University, Nicosia/Mersin 10, 99138, Turkey*
[6]*School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, P.R. China*

[*] Corresponding author: mubashar.arshad@edu.unige.it

Received: 07 May 2026 ***Revised:*** 23 June 2026 ***Accepted:*** 28 June 2026

Keywords: Enriched nonexpansiveness; HA-iteration; condition (I); Opial's condition; Banach space.

MSC 2020: 47H10, 46B45, 46B50, 34K06

Cite as: S. Asjad.; H. Ullah.; M. Arshad.; E. Hincal.; I. Ullah. *Convergence Analysis of HA-Iteration for Enriched Nonexpansive Mappings in Banach Space Settings*, Int. J. Appl. Math. Optim. AI., 1(1), (2026), 40–45.

Abstract

This paper analyzes the three-step HA-iteration scheme for enriched nonexpansive mappings. We establish weak convergence via Opial's condition and prove strong convergence under varying domain and mapping assumptions. A numerical example demonstrates that the HA-iteration converges more effectively than existing schemes for enriched nonexpansive mappings that are not classically nonexpansive. These results extend current literature and highlight the efficiency of the proposed method.

1. Introduction

Let E be a subset of a Banach space B, and $\mathcal{T} : E \to E$ be a self-map. $\mathcal{T}$ is *nonexpansive* if $\|\mathcal{T}w - \mathcal{T}w'\| \le \|w - w'\|$ for all $w, w' \in E$. Since Browder [1] and Gohde [2] established fixed point existence in uniformly convex Banach spaces (UCBS) [3], this theory has seen extensive development [4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

To broaden this framework, Berinde [14] introduced enriched nonexpansive mappings:

Definition 1.1. *A map $\mathcal{T} : E \to E$ is* enriched nonexpansive *if there exists $b \in [0, \infty)$ such that $\|b(w - w') + \mathcal{T}w - \mathcal{T}w'\| \le (b + 1)\|w - w'\|$ for all $w, w' \in E$.*

This class strictly includes nonexpansive mappings ($b = 0$). Given an enriched nonexpansive $\mathcal{T}$, we define the averaged operator $\mathcal{T}_\lambda w := (1 - \lambda)w + \lambda\mathcal{T}w$, where $\lambda = (b + 1)^{-1}$. It is known that $F_{\mathcal{T}_\lambda} = F_{\mathcal{T}}$.

While the Picard iteration [15] may fail for these mappings, several multi-step iterative schemes have been developed to ensure convergence, including those by Mann [16], Ishikawa [17], Noor [18], Agarwal [19], and Abbas [20]. We compare the proposed HA-iteration scheme against these methods and the HK-iteration [21]:

$$\begin{cases} u_i = (1 - \beta_i)w_i + \beta_i\mathcal{T}_\lambda^2 w_i, \\ v_i = (1 - \alpha_i)\mathcal{T}_\lambda^2 w_i + \alpha_i\mathcal{T}_\lambda^2 u_i, \\ w_{i+1} = \mathcal{T}_\lambda v_i, \end{cases} \tag{1}$$

where $\{\alpha_i\}, \{\beta_i\} \subset [0, 1]$. Our numerical analysis shows that the HA-iteration provides a faster rate of convergence than the aforementioned schemes. The remainder of this paper establishes formal convergence results for scheme (1) and provides validation.

2. Preliminaries

Let E be a closed convex subset of a UCBS B. For a bounded sequence $\{w_i\} \subset B$, we define the asymptotic radius and center relative to E as $r(E, \{w_i\}) := \inf_{w\in E} \limsup_{i\to\infty} \|w - w_i\|$ and $A(E, \{w_i\}) := \{w \in E : \limsup_{i\to\infty} \|w - w_i\| = r(E, \{w_i\})\}$. In a UCBS, the asymptotic center is a unique point [22, 23].

Definition 2.1 ([24]). *A Banach space B satisfies* Opial's property *if for any $\{w_i\}$ converging weakly to $w \in B$, $\limsup_{i\to\infty} \|w_i - w\| < \limsup_{i\to\infty} \|w_i - w'\|$ for all $w' \in B \setminus \{w\}$.*

Lemma 2.2. *Let $E \subseteq B$ and $\mathcal{T} : E \to E$.*

(i) *Nonexpansive mappings are enriched nonexpansive ($b = 0$).*

(ii) *If B satisfies Opial's property and $\mathcal{T}$ is nonexpansive, then $\{w_i\} \rightharpoonup w_0$ and $\lim_{i\to\infty} \|w_i - \mathcal{T}w_i\| = 0$ implies $w_0 \in F_{\mathcal{T}}$.*

Lemma 2.3 ([25]). *Let B be a UCBS and $0 < u \le \gamma_i \le v < 1$. If $\{v_i\}, \{w_i\} \subset B$ satisfy $\limsup_i \|v_i\| \le z$, $\limsup_i \|w_i\| \le z$, and $\lim_i \|\gamma_i v_i + (1 - \gamma_i)w_i\| = z$ for $z \ge 0$, then $\lim_i \|v_i - w_i\| = 0$.*

3. Main Results

Lemma 3.1. *Let E be a closed convex subset of a UCBS B, $T : E \to E$ enriched nonexpansive, and $F_T \neq \emptyset$. For the sequence $\{w_i\}$ generated by* (1)*, the limit $\lim_{i\to\infty} \|w_i - w_0\|$ exists for all $w_0 \in F_T$.*

Proof. Fix $w_0 \in F_T = F_{T_\lambda}$, where $T_\lambda = (1-\lambda)I + \lambda T$ with $\lambda = (1+b)^{-1}$. Since T is enriched nonexpansive, T_λ and T_λ^2 are nonexpansive. Since $T_\lambda w_0 = T_\lambda^2 w_0 = w_0$, we estimate:

$$\begin{aligned}
\|u_i - w_0\| &\leq (1-\beta_i)\|w_i - w_0\| + \beta_i\|w_i - w_0\| = \|w_i - w_0\|,\\
\|v_i - w_0\| &\leq (1-\alpha_i)\|w_i - w_0\| + \alpha_i\|u_i - w_0\| \leq \|w_i - w_0\|,\\
\|w_{i+1} - w_0\| &= \|T_\lambda v_i - T_\lambda w_0\| \leq \|v_i - w_0\| \leq \|w_i - w_0\|.
\end{aligned}$$

Thus, $\{\|w_i - w_0\|\}$ is nonincreasing and bounded, ensuring the limit exists. □

Lemma 3.2. *Let E be a closed convex subset of a UCBS B and $T : E \to E$ enriched nonexpansive. The sequence $\{w_i\}$ generated by* (1) *is bounded and satisfies $\lim_{i\to\infty} \|T_\lambda w_i - w_i\| = 0$ if and only if $F_T \neq \emptyset$.*

Proof. ($\Rightarrow$) With $w_0 \in F_T$, Lemma 3.1 implies $\{\|w_i - w_0\|\}$ converges to $z \geq 0$. Convexity of (1) yields $\limsup \|u_i - w_0\| \leq z$ and $\limsup \|v_i - w_0\| \leq z$. Since $\|w_{i+1} - w_0\| \leq \|v_i - w_0\|$, we find $\lim_{i\to\infty} \|v_i - w_0\| = z$. By Lemma 2.3 (Schu [25]), $\lim_{i\to\infty} \|T_\lambda^2 u_i - T_\lambda^2 w_i\| = 0$. Since T_λ is nonexpansive, $\|u_i - w_i\| \to 0$, implying $\|T_\lambda^2 w_i - w_i\| \to 0$. Finally, $\|T_\lambda w_i - w_i\| \leq \|T_\lambda w_i - T_\lambda^2 w_i\| + \|T_\lambda^2 w_i - w_i\| \to 0$.

($\Leftarrow$) Let $w_0 \in A(E, \{w_i\})$. Nonexpansiveness gives $r(T_\lambda w_0, \{w_i\}) \leq \limsup \|w_i - T_\lambda w_i\| + r(w_0, \{w_i\}) = r(w_0, \{w_i\})$. By uniqueness of the asymptotic center (Takahashi [23]), $T_\lambda w_0 = w_0$, so $F_T \neq \emptyset$. □

Theorem 3.3. *If E is a compact convex subset of a UCBS B, then $\{w_i\}$ converges strongly to $p \in F_T$.*

Proof. By Lemma 3.2, $\{w_i\}$ is bounded and $\lim \|w_i - T_\lambda w_i\| = 0$. Compactness of E ensures a subsequence $\{w_{i_r}\} \to \tilde{w} \in E$. Nonexpansiveness implies $\|w_{i_r} - T_\lambda \tilde{w}\| \to 0$, so $T_\lambda \tilde{w} = \tilde{w}$. Convergence of the full sequence follows from Lemma 3.1. □

Theorem 3.4. *If $\liminf_{i\to\infty} \operatorname{dist}(w_i, F_T) = 0$, then $\{w_i\}$ converges strongly to a fixed point.*

Proof. For any $p^* \in F_T$, Lemma 3.1 implies $\|w_n - p^*\| \leq \|w_{i_0} - p^*\|$. For $\varepsilon > 0$, choose i_0 such that $\operatorname{dist}(w_{i_0}, F_T) < \varepsilon/2$, hence $\|w_n - w_m\| < \varepsilon$ for $n, m \geq i_0$. The sequence is Cauchy and converges to some $\tilde{w} \in F_T$. □

Theorem 3.5. *If T_λ satisfies condition (I) (Senter and Dotson [26]), then $\{w_i\}$ converges strongly to a fixed point.*

Proof. Condition (I) gives $f(\operatorname{dist}(w_i, F_T)) \leq \|w_i - T_\lambda w_i\| \to 0$. Since $f(t) > 0$ for $t > 0$, $\operatorname{dist}(w_i, F_T) \to 0$. Convergence follows by Theorem 3.4. □

Theorem 3.6. *If B satisfies Opial's condition (Opial [24]), $\{w_i\}$ converges weakly to a fixed point.*

Proof. Since B is reflexive, $w_{i_s} \rightharpoonup p_1 \in F_T$. Assuming $w_{i_t} \rightharpoonup p_2 \neq p_1$, Opial's condition yields $\lim \|w_i - p_1\| < \lim \|w_i - p_2\|$ and vice versa, a contradiction. Thus, $w_i \rightharpoonup p_1$. □

4. Example

We illustrate the convergence behavior of the HA-iteration via a mapping that is enriched nonexpansive but fails to be classically nonexpansive.

Example 4.1. *Consider* $E = [1, 3] \subset \mathbb{R}$ *and* $Te = \frac{1}{2}(e + e^{-1})$*. First, we note* $T(E) \subset [2/3, 2] \subset E$*, ensuring the iteration is well-defined. Regarding nonexpansiveness, for* $e = 3, e' = 1$*, we have* $|T(3) - T(1)| = 2/3$*. While* $|e - e'| = 2$*, for other points, the ratio* $|T(e) - T(e')|/|e - e'|$ *can exceed 1, confirming* T *is not classically nonexpansive.*

Enriched Nonexpansiveness: *For* $b = 1$ *and* $e, e' \in [2/3, 2]$*, we evaluate the condition:*

$$|1(e - e') + Te - Te'| = |e - e'| \left| \frac{3}{2} - \frac{1}{2ee'} \right|.$$

Since $ee' \geq 4/9$*, it follows that* $|\frac{3}{2} - \frac{1}{2ee'}| \leq 2 = b + 1$*. Thus,* T *is enriched nonexpansive with* $b = 1$*.*

Convergence: *With* $\lambda = (b+1)^{-1} = 1/2$*, the averaged operator is* $T_\lambda e = \frac{3}{4}e + \frac{1}{4e}$*. The fixed point* $Te = e$ *yields* $e = 1$*. Starting from* $w_1 = 2.5$*, the HA-iteration sequence* w_i *converges rapidly to the fixed point* 1*, as shown in Table 1 and Figure 1.*

Table 1: Comparison of the proposed HA-iteration with existing iterative schemes.

k	HA	HK	Agarwal	Abbas	Picard-Noor
1	2.5	2.5	2.5	2.5	2.5
2	1.0018118940	1.0477373725	1.4253638030	1.1391793728	1.1391793730
3	1	1.0000003600	1.0585738690	1.0015183890	1.00151838900
4	1	1	1.0014658780	1.0000001960	1.00000019600
5	1	1	1.0000009660	1	1
6	1	1	1	1	1

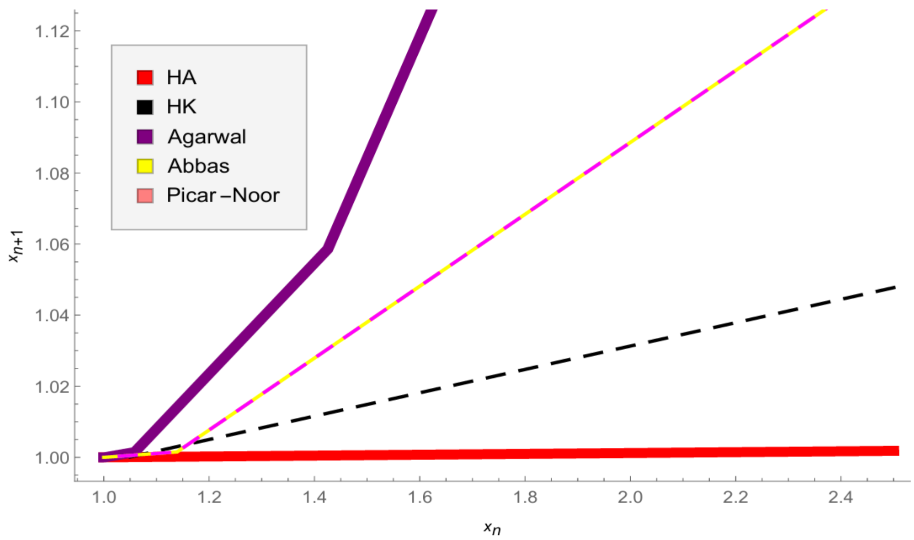

Figure 1: Graphical comparison of HA, HK, Agarwal, Abbas and Picard–Noor competing iteration schemes converging to the unique fixed point $\mathcal{T}$ in Example 4.1.

5. Conclusion

In this study, we introduced the HA-iteration process for approximating fixed points of enriched nonexpansive mappings in Banach spaces. We established both weak convergence, under Opial's condition, and strong convergence, utilizing compactness arguments, condition (I), and distance criteria.

These results extend existing fixed point theorems for classical nonexpansive mappings to the broader class of enriched nonexpansive operators. Our theoretical analysis confirms the stability of the proposed scheme, while comparative numerical investigations demonstrate that the HA-iteration achieves a faster rate of convergence than the HK, Agarwal, Abbas, and Picard–Noor processes.

Given its computational efficiency, the HA-iteration provides a robust framework for addressing fixed point problems and offers a promising tool for further developments in nonlinear analysis and applied fixed point theory.

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Artificial Intelligence Statement: No artificial intelligence tools were used in the preparation of this manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by authors.

References

[1] F. E. Browder, *Nonexpansive Nonlinear Operators in a Banach Space*, Proc. Natl. Acad. Sci. USA, 54, (1965), 1041–1044.

[2] D. Gohde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr., 30, (1965), 251–258.

[3] J. A. Clarkson, *Uniformly Convex Spaces*, Trans. Amer. Math. Soc., 40, (1936), 396–414.

[4] K. Goebel.; W. A. Kirk, *A Fixed Point Theorem for Asymptotically Nonexpansive Mappings*, Proc. Amer. Math. Soc., 35, (1972), 171–174.

[5] K. Aoyama.; F. Kohsaka, *Fixed Point Theorem for α-Nonexpansive Mappings in Banach Spaces*, Nonlinear Anal., 74, (2011), 4387–4391.

[6] J. S. Bae, *Fixed Point Theorems of Generalized Nonexpansive Mappings*, J. Korean Math. Soc., 21, (1984), 233–248.

[7] J. Bogin, *A Generalization of a Fixed Point Theorem of Goebel, Kirk and Shimi*, Canad. Math. Bull., 19, (1976), 7–12.

[8] K. Goebel.; W. A. Kirk.; T. N. Shimi, *A Fixed Point Theorem in Uniformly Convex Spaces*, Boll. Unione Mat. Ital., 7, (1973), 67–75.

[9] E. Llorens-Fuster.; E. Moreno-Gálvez, *The Fixed Point Theory for Some Generalized Nonexpansive Mappings*, Abstr. Appl. Anal., 2011, (2011), 1–15.

[10] T. Suzuki, *Fixed Point Theorems and Convergence Theorems for Some Generalized Nonexpansive Mappings*, J. Math. Anal. Appl., 340, (2008), 1088–1095.

[11] R. Pant.; R. Shukla, *Approximating Fixed Points of Generalized α-Nonexpansive Mappings in Banach Spaces*, Numer. Funct. Anal. Optim., 38, (2017), 248–266.

[12] J. Garcia-Falset.; E. Llorens-Fuster.; T. Suzuki, *Fixed Point Theory for a Class of Generalized Nonexpansive Mappings*, J. Math. Anal. Appl., 375, (2011), 185–195.

[13] B. Patir.; N. Goswami.; V. N. Mishra, *Some Results on Fixed Point Theory for a Class of Generalized Nonexpansive Mappings*, Fixed Point Theory Appl., (2018).

[14] V. Berinde, *Approximating Fixed Points of Enriched Nonexpansive Mappings by Krasnosel'skiĭ Iteration in Hilbert Spaces*, Carpathian J. Math., 35, (2019), 293–304.

[15] E. M. Picard, *Mémoire sur la Théorie des Équations aux Dérivées Partielles et la Méthode des Approximations Successives*, J. Math. Pure Appl., 6, (1890), 145–210.

[16] W. R. Mann, *Mean Value Methods in Iteration*, Proc. Amer. Math. Soc., 4, (1953), 506–510.

[17] S. Ishikawa, *Fixed Points by a New Iteration Method*, Proc. Amer. Math. Soc., 44, (1974), 147–150.

[18] M. A. Noor, *New Approximation Schemes for General Variational Inequalities*, J. Math. Anal. Appl., 251, (2000), 217–229.

[19] R. P. Agarwal.; D. O'Regan.; D. R. Sahu, *Iterative Construction of Fixed Points of Nearly Asymptotically Non-Expansive Mappings*, J. Nonlinear Convex Anal., 8, (2007), 61–79.

[20] M. Abbas.; T. Nazir, *A New Faster Iteration Process Applied to Constrained Minimization and Feasibility Problems*, Math. Vesnik, 66, (2014), 223–234.

[21] H. Ullah.; K. Ullah.; A. Asghar.; M. Tariq.; H. Ahmad.; M. F. Abouelenein, *Some New Convergence Approaches in Banach Spaces Involving the HK-Iteration Method with Application*, (2025).

[22] R. P. Agarwal.; D. O'Regan.; D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Springer, New York, (2009).

[23] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, (2000).

[24] Z. Opial, *Weak and Strong Convergence of the Sequence of Successive Approximations for Non-Expansive Mappings*, Bull. Amer. Math. Soc., 73, (1967), 591–595.

[25] J. Schu, *Weak and Strong Convergence to Fixed Points of Asymptotically Nonexpansive Mappings*, Bull. Austral. Math. Soc., 43, (1991), 153–159.

[26] H. F. Senter.; W. G. Dotson, *Approximating Fixed Points of Non-Expansive Mappings*, Proc. Amer. Math. Soc., 44, (1974), 375–380.

INTELLECTA PRIME
PUBLISHER

License

This work is licensed under a Creative Commons Attribution 4.0 International License.

CC BY