



Research Article

Midpoint Type Inequality for Multiplicatively Strong Convex Function via Multiplicative Calculus

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Received: 12 May 2026 **Revised:** 26 June 2026 **Accepted:** 27 June 2026

Keywords: Convex function; Multiplicative calculus; Midpoint type inequality; Multiplicatively strong convex function.

MSC 2020: 05A30; 26A51; 26D10; 26D15.

Cite as: D. Khan.; I. Khan.; J.E.N. Valdes.; W.M. Abdelfattah.; M. Azeem. *Midpoint type inequality for multiplicatively strong convex function via multiplicative calculus*, Int. J. Appl. Math. Optim. AI, 1(1), (2026), 31–39.

Abstract

In this paper, we first establish a new identity for multiplicatively differentiable functions within the framework of multiplicative calculus. By exploiting this identity, we derive a novel midpoint-type inequality for multiplicatively strong convex functions. The obtained result extends the existing theory of multiplicative integral inequalities and provides a unified approach for investigating midpoint estimates in multiplicative calculus.

1. Introduction

First introduced by Karamardian in [1], strongly convex functions are an advanced version of standard convex functions. These functions have extra qualities that make them particularly useful in certain mathematical scenarios. Polyak established the concept of strongly convex functions in his work [2], as several scholars have pointed out. This suggests that Polyak, who may have been the first to formalize and explore the idea, made a significant contribution to the creation of the mathematical concept of strongly convex functions. Lin and Fukushima expanded the traditional concept of convexity by introducing the idea of higher-order highly convex functions [3]. They demonstrated that if and only if a function's gradient exhibits strong monotonicity of the same order, it is only higher-order highly convex. This is valid for any functions that are continuously differentiable. In 2011, Srivastava and colleagues added to the corpus of work on mathematical inequalities by refining and extending two significant inequalities, the Jensen and H-H (Hermite-Hadamard) inequalities. Their work [4] increased the precision and use of these basic mathematical tools by focusing on these inequalities within the context of n variables. Mishra and Sharma initially suggested strongly generalized convex functions of higher order in [5]. By considering functions with more complex structures and behaviors, this sophisticated term improves the traditional concept of convexity, particularly when analyzed at a higher degree. Based on this concept, they created new integral inequalities of the H-H type. Because these inequalities are specifically made for the class of strongly generalized convex functions of higher order, they provide more advanced and powerful tools for mathematical study within this framework. Noor and Noor defined and presented the idea of strongly exponentially convex functions in [6], which was a significant addition. The idea of classical convexity has been extended to allow for more complex functions and a greater variety of applications by adding the concept of exponential components. Butt found Jensen-H-H Mercer type inequalities on fractal space using the concept of strongly convex functions [7]. This generalization brought together the relatively new science of fractals and the notion of strongly convex functions. As a result, the study of inequalities inside atypical geometrical spaces saw the emergence of new research opportunities. Merentes and Nikodem [8] contributed some helpful information regarding strongly convex functions.

After the work of Ali et al. [9], the theory of multiplicative calculus has received a lot of attention and has proven to be highly helpful in the study of integral inequalities. Many multiplicative integer-order inequalities have been developed for various kinds of functions. These include multiplicative harmonic convexity [10] and multiplicative preinvex P -convexity [11]. Additionally, researchers have examined various types of multiplicative integer-order inequalities. For example, Khan and Budak [12] constructed H-H type inequalities for $*$ -differentiable functions in the context of multiplicative integrals, and Xie et al. [13] studied analogous inequalities for $**$ -differentiable functions. For additional research on integer-order inequalities, such as those of the Ostrowski, Simpson, and Maclaurin types, see [14, 15] and related sources. The reader is recommended to [16, 17, 18, 19, 20, 21, 22] for more details on recent developments in multiplicative fractional calculus that go much beyond integral inequalities.

Motivated by recent developments in multiplicative calculus and multiplicative integral inequalities, we establish a new identity for multiplicatively differentiable functions. Using this identity, we derive a novel midpoint-type inequality for multiplicatively strong convex functions, thereby extending the existing theory of multiplicative integral inequalities.

2. Material and Methods

In this section, we throw light on the basic notions and properties associated with multiplicative calculus.

Definition 2.1. [8] A function Φ is termed as strongly convex on I , if

$$\Phi(\vartheta_1 t + (1-t)\vartheta_2) \leq t\Phi(\vartheta_1) + (1-t)\Phi(\vartheta_2) - ct(1-t)(\vartheta_1 - \vartheta_2)^2,$$

holds $\forall \vartheta_1, \vartheta_2 \in I, t \in [0, 1]$ and $c > 0$. Where I is a convex set.

Definition 2.2. [6] A function Φ is termed as multiplicatively strong convex on I , if

$$\log \Phi(\vartheta_1 t + (1-t)\vartheta_2) \leq t \log \Phi(\vartheta_1) + (1-t) \log \Phi(\vartheta_2) - ct(1-t)(\vartheta_1 - \vartheta_2)^2, \quad (1)$$

holds $\forall \vartheta_1, \vartheta_2 \in I, t \in [0, 1]$ and $c > 0$.

An alternate way of expression 1 is given by

$$\Phi(\vartheta_1 t + (1-t)\vartheta_2) \leq \frac{[\Phi(\vartheta_1)]^t [\Phi(\vartheta_2)]^{1-t}}{\exp\{ct(1-t)(\vartheta_1 - \vartheta_2)^2\}},$$

$\forall \vartheta_1, \vartheta_2 \in I$ and $t \in [0, 1]$.

Next consider the properties associated with multiplicative calculus

Definition 2.3. [17] The $*$ derivative for a positive valued function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is defined as in the subsequent

$$\frac{d^* \Phi}{d\mathfrak{x}} = \Phi^*(\mathfrak{x}) = \lim_{h \rightarrow 0} \left(\frac{\Phi(\mathfrak{x} + h)}{\Phi(\mathfrak{x})} \right)^{\frac{1}{h}}.$$

Remark 2.4. The relationship between the classical and $*$ derivatives of a positive function is as follows:

$$\Phi^*(\mathfrak{x}) = \exp\{(\ln \Phi(\mathfrak{x}))'\} = \exp\left\{\frac{\Phi'(\mathfrak{x})}{\Phi(\mathfrak{x})}\right\}.$$

While the relationship between $*$ integral and the Riemann integral is as follows [17]:

Proposition 2.5. Let the function Φ is both Riemann and $*$ integrable on the $[\vartheta_1, \vartheta_2]$ then.

$$\int_{\vartheta_2}^{\vartheta_1} (\Phi(\mathfrak{x}))^{d\mathfrak{x}} = \exp\left\{\int_{\vartheta_2}^{\vartheta_1} \ln(\Phi(\mathfrak{x})) d\mathfrak{x}\right\}.$$

Moreover, Bashirov et al. [17] established several properties and related results concerning $*$ integrable functions, including the following:

Proposition 2.6. Let the function Φ is $*$ integrable on $[\mu_0, \nu_0]$, then

1. $\int_{\vartheta_1}^{\vartheta_2} ((\Phi(\mathfrak{x}))^p)^{d\mathfrak{x}} = \int_{\vartheta_2}^{\vartheta_1} ((\Phi(\mathfrak{x}))^{d\mathfrak{x}})^p$
2. $\int_{\vartheta_1}^{\vartheta_2} (\Phi(\mathfrak{x}) \Upsilon(\mathfrak{x}))^{d\mathfrak{x}} = \int_{\vartheta_1}^{\vartheta_2} (\Phi(\mathfrak{x}))^{d\mathfrak{x}} \cdot \int_{\vartheta_1}^{\vartheta_2} (\Upsilon(\mathfrak{x}))^{d\mathfrak{x}}$

3. $\int_{\vartheta_1}^{\vartheta_2} \left(\frac{\Phi(\mathfrak{x})}{\Upsilon(\mathfrak{x})} \right)^{\mathrm{d}\mathfrak{x}} = \frac{\int_{\vartheta_1}^{\vartheta_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}}{\int_{\vartheta_1}^{\vartheta_2} (\Upsilon(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}}$
4. $\int_{\vartheta_1}^{\vartheta_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}} = \int_{\vartheta_1}^c (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}} \cdot \int_c^{\vartheta_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}, \quad \vartheta_1 \leq c \leq \vartheta_2$
5. $\int_{\vartheta_1}^{\vartheta_1} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}} = 1, \quad \int_{\vartheta_1}^{\vartheta_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}} = \left(\int_{\vartheta_2}^{\vartheta_1} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}} \right)^{-1}.$

Theorem 2.7. (Multiplicative version of the integration by parts formula [17]) Let the functions $\Phi : [\vartheta_1, \vartheta_2] \rightarrow \mathbb{R}$ and $\Upsilon : [\vartheta_1, \vartheta_2] \rightarrow \mathbb{R}$ be $*$ -differentiable and differentiable respectively and the function Φ^Υ be a $*$ -integrable then

$$\int_{\vartheta_1}^{\vartheta_2} \left(\Phi^*(\mathfrak{x})^{\Upsilon(\mathfrak{x})} \right)^{\mathrm{d}\mathfrak{x}} = \frac{\Phi(\vartheta_2)^{\Upsilon(\vartheta_2)}}{\Phi(\vartheta_1)^{\Upsilon(\vartheta_1)}} \times \frac{1}{\int_{\vartheta_1}^{\vartheta_2} \left(\Phi(\mathfrak{x})^{\Upsilon'(\mathfrak{x})} \right)^{\mathrm{d}\mathfrak{x}}}.$$

Lemma 2.8. [17] Let $\Phi : [\vartheta_1, \vartheta_2] \rightarrow \mathbb{R}$ and $\Upsilon : [\vartheta_1, \vartheta_2] \rightarrow \mathbb{R}$ be $*$ -differentiable and differentiable respectively, and Φ^Υ is $*$ -integrable then

$$\int_{\vartheta_1}^{\vartheta_2} \left(\Phi^*(h(\mathfrak{x}))^{h'(\mathfrak{x})\Upsilon(\mathfrak{x})} \right)^{\mathrm{d}\mathfrak{x}} = \frac{\Phi(\vartheta_2)^{\Upsilon(\vartheta_2)}}{\Phi(\vartheta_1)^{\Upsilon(\vartheta_1)}} \times \frac{1}{\int_{\vartheta_1}^{\vartheta_2} \left(\Phi(h(\mathfrak{x}))^{\Upsilon'(\mathfrak{x})} \right)^{\mathrm{d}\mathfrak{x}}}.$$

3. Main Results

In this section, we first develop the identity, then utilizing that identity, we establish the midpoint type inequality.

Lemma 3.1. Let $\Phi : [\vartheta_1, \vartheta_2] \rightarrow \mathbb{R}^+$ be a multiplicative differentiable mapping on $[\vartheta_1, \vartheta_2]$ with $\vartheta_1 < \vartheta_2$. If Φ^* is multiplicative integrable on $[\vartheta_1, \vartheta_2]$, then we have the following identity for multiplicative integrals:

$$\begin{aligned} \Phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \left(\int_{\vartheta_1}^{\vartheta_2} \Phi(\mathfrak{x})^{\mathrm{d}\mathfrak{x}} \right)^{\frac{1}{\vartheta_1 - \vartheta_2}} &= \left(\int_0^1 \left(\left(\Phi^* \left((1-t)\vartheta_1 + t\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^t \right)^{\mathrm{d}t} \right)^{\frac{\vartheta_2 - \vartheta_1}{4}} \\ &\times \left(\int_0^1 \left(\left(\Phi^* \left((1-t)\frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right)^{t-1} \right)^{\mathrm{d}t} \right)^{\frac{\vartheta_2 - \vartheta_1}{4}}. \end{aligned} \quad (2)$$

Proof. Let

$$I_1 = \left(\int_0^1 \left(\left(\Phi^* \left((1-t)\vartheta_1 + t\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^t \right)^{\mathrm{d}t} \right)^{\frac{\vartheta_2 - \vartheta_1}{4}},$$

and

$$I_2 = \left(\int_0^1 \left(\left(\Phi^* \left((1-t)\frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right)^{t-1} \right)^{\mathrm{d}t} \right)^{\frac{\vartheta_2 - \vartheta_1}{4}}.$$

Using the integration by parts for multiplicative integrals, from I_1 we have

$$\begin{aligned}
 I_1 &= \left(\int_0^1 \left(\left(\Phi^* \left((1-t)\vartheta_1 + t \frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^t \right)^{\frac{\vartheta_2 - \vartheta_1}{4}} dt \right) \\
 &= \int_0^1 \left(\left(\Phi^* \left((1-t)\vartheta_1 + t \frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{\vartheta_2 - \vartheta_1}{4} t} \right)^{dt} \\
 &= \int_0^1 \left(\left(\Phi^* \left((1-t)\vartheta_1 + t \frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\left(\frac{\vartheta_2 - \vartheta_1}{2} \right)^{\frac{1}{2}} t} \right)^{dt} \\
 &= \frac{\left(\Phi \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{1}{2}}}{1} \cdot \frac{1}{\int_0^1 \left(\left(\Phi \left((1-t) \frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right)^{\frac{1}{2}} \right)^{dt}} \\
 &= \left(\Phi \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{1}{2}} \frac{1}{\left(\int_0^1 \left(\left(\Phi \left((1-t) \frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right)^{\frac{1}{2}} \right)^{dt} \right)} \\
 &= \left(\Phi \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{1}{2}} \left(\int_{\vartheta_1}^{\frac{\vartheta_1 + \vartheta_2}{2}} \Phi(r) dr \right)^{\frac{1}{\vartheta_1 - \vartheta_2}}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 I_2 &= \left(\int_0^1 \left(\left(\Phi^* \left((1-t) \frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right)^{t-1} \right)^{\frac{\vartheta_2 - \vartheta_1}{4}} dt \right) \\
 &= \int_0^1 \left(\left(\Phi^* \left((1-t) \frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right)^{\frac{\vartheta_2 - \vartheta_1}{4} (t-1)} \right)^{dt} \\
 &= \frac{1}{\left(\Phi \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{1}{2}}} \cdot \frac{1}{\int_0^1 \left(\left(\Phi \left((1-t) \frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right) \right)^{dt}} \\
 &= \left(\Phi \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{1}{2}} \frac{1}{\int_0^1 \left(\left(\Phi \left((1-t) \frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) \right)^{\frac{1}{2}} \right)^{dt}} \\
 &= \left(\Phi \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{1}{2}} \left(\int_0^1 \Phi \left((1-t) \frac{\vartheta_1 + \vartheta_2}{2} + t\vartheta_2 \right) dt \right)^{-\frac{1}{2}} \\
 &= \left(\Phi \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right)^{\frac{1}{2}} \left(\int_{\frac{\vartheta_1 + \vartheta_2}{2}}^{\vartheta_2} \Phi(r) dr \right)^{\frac{1}{\vartheta_1 - \vartheta_2}}.
 \end{aligned}$$

Taking the product of the above identities, we obtain the required result. Hence, the proof is complete. $\square$

Theorem 3.2. Let the function $\Phi : [\vartheta_1, \vartheta_2] \longrightarrow \mathbb{R}^+$ be a multiplicatively differentiable on $[\vartheta_1, \vartheta_2]$ with $\vartheta_1 < \vartheta_2$. If Φ^* is multiplicatively strong convex function on $[\vartheta_1, \vartheta_2]$, then

$$\left| \Phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \left(\int_{\vartheta_1}^{\vartheta_2} (\Phi(x))^{\mathrm{d}x} \right)^{\frac{1}{\vartheta_1 - \vartheta_2}} \right| \leq \frac{\left[\Phi^*(\vartheta_1) \Phi^*(\vartheta_2) \left(\Phi^*\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \right)^4 \right]^{\frac{\vartheta_2 - \vartheta_1}{24}}}{\exp\left\{ \frac{(\vartheta_2 - \vartheta_1)^3 c}{96} \right\}}, \quad (3)$$

holds for all $c > 0$.

Proof. Consider the Lemma 3.1, and taking absolute values on both sides yields

$$\begin{aligned} & \left| \Phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \left(\int_{\vartheta_1}^{\vartheta_2} (\Phi(x))^{\mathrm{d}x} \right)^{\frac{1}{\vartheta_1 - \vartheta_2}} \right| \\ & \leq \left(\exp \left\{ \frac{\vartheta_2 - \vartheta_1}{4} \int_0^1 \left| \ln \left(\Phi^* \left((1-t)\vartheta_1 + t \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right) \right) \right| \mathrm{d}t \right\} \right) \\ & \times \left(\exp \left\{ \frac{\vartheta_2 - \vartheta_1}{4} \int_0^1 \left| \ln \left(\Phi^* \left((1-t) \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) + t\vartheta_2 \right) \right) \right| \mathrm{d}t \right\} \right) \\ & = \left(\exp \left\{ \frac{\vartheta_2 - \vartheta_1}{4} \int_0^1 t \left| \ln \left(\Phi^* \left((1-t)\vartheta_1 + t \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right) \right) \right| \mathrm{d}t \right\} \right) \\ & \times \left(\exp \left\{ \frac{\vartheta_2 - \vartheta_1}{4} \int_0^1 (1-t) \left| \ln \left(\Phi^* \left((1-t) \left(\frac{\vartheta_1 + \vartheta_2}{2} \right) + t\vartheta_2 \right) \right) \right| \mathrm{d}t \right\} \right) \\ & \leq \left(\exp \left\{ \frac{\vartheta_2 - \vartheta_1}{4} \int_0^1 t \ln \left(\frac{(\Phi^*(\vartheta_1))^{1-t} (\Phi^*\left(\frac{\vartheta_1 + \vartheta_2}{2}\right))^t}{\exp\{ct(1-t)(\frac{\vartheta_2 - \vartheta_1}{2})^2\}} \right) \mathrm{d}t \right\} \right) \\ & \times \left(\exp \left\{ \frac{\vartheta_2 - \vartheta_1}{4} \int_0^1 (1-t) \ln \left(\frac{(\Phi^*\left(\frac{\vartheta_1 + \vartheta_2}{2}\right))^{1-t} (\Phi^*(\vartheta_2))^t}{\exp\{ct(1-t)(\frac{\vartheta_2 - \vartheta_1}{2})^2\}} \right) \mathrm{d}t \right\} \right) \\ & = \exp \left\{ \left(\frac{\vartheta_2 - \vartheta_1}{4} \right) \int_0^1 \left(t \left[(1-t) \ln \Phi^*(\vartheta_1) + t \ln \Phi^*\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) - c t(1-t) \left(\frac{\vartheta_2 - \vartheta_1}{2} \right)^2 \right] \right. \right. \\ & \quad \left. \left. + (1-t) \left[(1-t) \ln \Phi^*\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + t \ln \Phi^*(\vartheta_2) - c t(1-t) \left(\frac{\vartheta_2 - \vartheta_1}{2} \right)^2 \right] \right) \mathrm{d}t \right\} \\ & = \frac{\left[(\Phi^*(\vartheta_1) \Phi^*(\vartheta_2))^4 \left(\Phi^*\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \right)^{16} \right]^{\frac{\vartheta_2 - \vartheta_1}{96}}}{\exp\left\{ \frac{(\vartheta_2 - \vartheta_1)^3 c}{96} \right\}} = \frac{\left[\Phi^*(\vartheta_1) \Phi^*(\vartheta_2) \left(\Phi^*\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \right)^4 \right]^{\frac{\vartheta_2 - \vartheta_1}{24}}}{\exp\left\{ \frac{(\vartheta_2 - \vartheta_1)^3 c}{96} \right\}}. \end{aligned}$$

□

Remark 3.3. If we set $c=0$ in the Theorem 3.2, we get the midpoint type inequality for multiplicatively convex function.

Remark 3.4. If we set $\Phi^* \leq M$ in the Theorem 3.2, we attain the below mentioned result.

$$\left| \Phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \left(\int_{\vartheta_1}^{\vartheta_2} (\Phi(x))^{\mathrm{d}x} \right)^{\frac{1}{\vartheta_1 - \vartheta_2}} \right| \exp\left\{ \frac{(\vartheta_2 - \vartheta_1)^3 c}{96} \right\} \leq [M]^{\frac{\vartheta_2 - \vartheta_1}{4}}.$$

Remark 3.5. If we apply the definition of multiplicatively strong convex function on $\Phi^*\left(\frac{\vartheta_1+\vartheta_2}{2}\right)$, then we attain

$$\left| \Phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \left(\int_{\vartheta_1}^{\vartheta_2} (\Phi(x))^{\mathrm{d}x} \right)^{\frac{1}{\vartheta_1 - \vartheta_2}} \right| \leq \frac{[\Phi^*(\vartheta_1)\Phi^*(\vartheta_2)]^{\frac{\vartheta_2 - \vartheta_1}{8}}}{\exp\left\{\frac{5c(\vartheta_2 - \vartheta_1)^3}{96}\right\}}.$$

4. Applications

This section is devoted to the applications of the obtained results in terms of special means. Let us consider some means for the numbers ϑ_1 and ϑ_2 .

The Arithmetic mean: $A(\vartheta_1, \vartheta_2) = \frac{\vartheta_1 + \vartheta_2}{2}$.

The logarithmic means: $L(\vartheta_1, \vartheta_2) = \frac{\vartheta_2 - \vartheta_1}{\ln \vartheta_2 - \ln \vartheta_1}$, $\vartheta_1, \vartheta_2 > 0$ and $\vartheta_1 \neq \vartheta_2$.

The m-Logarithmic mean: $L_m(\vartheta_1, \vartheta_2) = \left(\frac{\vartheta_2^{m+1} - \vartheta_1^{m+1}}{(m+1)(\vartheta_2 - \vartheta_1)} \right)^{\frac{1}{m}}$,

$\vartheta_1, \vartheta_2 > 0$, $\vartheta_1 \neq \vartheta_2$ and $m \in \mathbb{R} \setminus \{-1, 0\}$.

The function $f(x) = x^m$, is strongly convex with modulus $c=1$ and $m \geq 2$. Therefore the function $f(x) = \exp\{x^m\}$ is then multiplicatively strong convex. Also the function $f^*(x) = \exp\{mx^{m-1}\}$ is multiplicatively strong convex with modulus $c=1$ and $m \geq 3$. Setting these functions in Theorem 3.2, we achieve the following recurrence relation.

Proposition 4.1. Let $\vartheta_1, \vartheta_2 \in \mathbb{R}^+$ with $0 < \vartheta_1 < \vartheta_2$, then

$$\begin{aligned} & \exp\{A^m(\vartheta_1, \vartheta_2) - L_m^m(\vartheta_1, \vartheta_2)\} \\ & \leq \exp\left\{[2mA(\vartheta_1^{m-1}, \vartheta_2^{m-1}) + 4mA(\vartheta_1, \vartheta_2)]\left(\frac{\vartheta_2 - \vartheta_1}{2}\right) - \frac{c(\vartheta_2 - \vartheta_1)^3}{96}\right\}. \end{aligned} \quad (4)$$

Proof. The result 4 is determined by employing the functions $f(x) = \exp\{x^m\}$ and $f^*(x) = \exp\{mx^{m-1}\}$ in Theorem 3.2. $\square$

5. Conclusion

In this paper, we established a new identity for multiplicatively differentiable functions in the setting of multiplicative calculus and employed it to derive a novel midpoint-type inequality for multiplicatively strong convex functions. The obtained inequality extends the existing literature on multiplicative integral inequalities by incorporating the concept of multiplicative strong convexity into the midpoint framework. The proposed identity provides a simple and effective analytical tool for investigating multiplicative integral estimates and may serve as a foundation for deriving further Hermite–Hadamard-, trapezoid-, Simpson-, Ostrowski-, Fejér-, and Bullen-type inequalities in multiplicative calculus. Moreover, the developed approach can be extended to other generalized classes of multiplicatively convex functions and to fractional operators in multiplicative calculus, offering several promising directions for future research.

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Artificial Intelligence Statement: No artificial intelligence tools were used in the preparation of this manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by authors.

References

- [1] S. Karamardian. *The nonlinear complementarity problem with applications, Part II*, J. Optim. Theory Appl., 4(3), (1969), 167–181.
- [2] B.T. Polyak. *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Sov. Math. Dokl., 7, (1966), 72–75.
- [3] G.H. Lin.; M. Fukushima. *Some exact penalty results for nonlinear programs and mathematical programs with equilibrium constraints*, J. Optim. Theory Appl., 118(1), (2003), 67–80.
- [4] H.M. Srivastava.; Z.H. Zhang.; Y.D. Wu. *Some further refinements and extensions of the Hermite–Hadamard and Jensen inequalities in several variables*, Math. Comput. Model., 54(11–12), (2011), 2709–2717.
- [5] S.K. Mishra.; N. Sharma. *On strongly generalized convex functions of higher order*, Math. Inequal. Appl., 22(1), (2019), 111–121.
- [6] M.A. Noor.; K.I. Noor. *Strongly exponentially convex functions*, UPB Sci. Bull., Ser. A, Appl. Math. Phys., 81(4), (2019), 75–84.
- [7] S.I. Butt. *Generalized Jensen–Hermite–Hadamard–Mercer type inequalities for generalized strongly convex functions on fractal sets*, Turkish Journal of Science, 8(2), (2024), 51–63.
- [8] N. Merentes.; K. Nikodem. *Remarks on strongly convex functions*, Aequ. Math., 80(1–2), (2010), 193–199.
- [9] M.A. Ali.; M. Abbas.; Z. Zhang.; I. B. Sial.; R. Arif. *On integral inequalities for product and quotient of two multiplicatively convex functions*, Asian Res. J. Math., 12(3), (2019), 1–11.
- [10] S. Özcan.; S.I. Butt. *Hermite–Hadamard type inequalities for multiplicatively harmonic convex functions*, J. Inequal. Appl., 2023(1), (2023), 120.
- [11] H. Kadakal.; M. Kadakal. *Multiplicatively preinvex P -functions*, J. Sci. Arts, 23(1), (2023), 21–32.
- [12] S. Khan.; H. Budak. *On midpoint and trapezoid type inequalities for multiplicative integrals*, Mathematica, 64(87), (2022), 95–108.
- [13] J.Q. Xie.; M.A. Ali.; T. Sitthiwiratttham. *Some new midpoint and trapezoidal type inequalities in multiplicative calculus with applications*, Filomat, 37(20), (2023), 6665–6675.
- [14] B. Meftah. *Maclaurin type inequalities for multiplicatively convex functions*, Proc. Amer. Math. Soc., 151(5), (2023), 2115–2125.
- [15] M.A. Ali.; H. Budak.; M.Z. Sarikaya.; Z. Y. Zhang. *Ostrowski and Simpson type inequalities for multiplicative integrals*, Proyecciones, 40(3), (2021), 743–763.

- [16] T. Abdeljawad.; M. Grossman, *On geometric fractional calculus*, J. Semigroup Theory Appl., 2016(2016) ID 2, 14 p.
- [17] A.E. Bashirov.; E.M. Kurpinar, A. Özyapici, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., 337(1)(2008), 36-48.
- [18] S.I. Butt.; D. Khan.; V.D. Breaz. *Fractional integral inequalities for strongly convex functions via multiplicative calculus with applications*, Bound. Value Probl., 2025, (2025), 102.
- [19] S.I. Butt.; D. Khan. *Superquadraticity via multiplicative calculus*, J. Comput. Appl. Math., 485, (2026), 117528.
- [20] D. Khan.; S.I. Butt. *Multiplicatively h -superquadratic functions and their applications in probability, special functions, and means*, Math. Methods Appl. Sci., (2026), 1–36.
- [21] M. Tariq.; W. Afzal.; M. Nadeem.; A.E. Munoz-Zavala.; J.E. Mac. *Novel fractional Hermite–Hadamard and product-type inequalities via Raina function and preinvex mappings with entropy applications*, Eur. J. Pure Appl. Math., 18(3) (2025), 6556–6556.
- [22] H. Ahmad.; M. Tariq.; A. Asghar.; W. Afzal.; M. Aphane.; R. Efendiev. *Some new notions of mathematical integral inequalities: theory and applications*, Int. J. Anal. Appl., 24 (2026), 175.



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