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Research Article

Some Results Within the Framework of Conformable Fractio--nal Calculus

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Abstract

In this paper, we employ fractional conformable integral operators to develop novel versions of Pachpatte-type integral inequalities for functions defined on harmonic sets, namely for harmonic h-Godunova–Levin mappings. We provide nontrivial numerical examples accompanied by graphical illustrations for different values of the fractional parameters to show the validity and application of the derived results.

1. Introduction

Due to its numerous applications in mathematics and physics, mathematical inequalities are crucial to the study of mathematics as well as other branches of mathematics. The convex function, which is defined as follows, is one of the most important functions used to explore intriguing inequalities: Let $\mathbb{I} \subset \mathbb{R}$ be a non-empty interval. The function $\zeta : \mathbb{I} \to \mathbb{R}$ is called convex if

$$\zeta(\mathfrak{d}_1\mathfrak{z}_1 + (1-\mathfrak{d}_1)\mathfrak{z}_2) \leq \mathfrak{d}_1\zeta(\mathfrak{z}_1) + (1-\mathfrak{d}_1)\zeta(\mathfrak{z}_2),$$

holds for every $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{I}$ and $\mathfrak{d}_1 \in [0,1]$. Convex functions and, in particular, one of the well-known inequalities for convex functions, the Pachpatte-type inequality [1], which is described in the following theorem, have recently captivated a lot of academics. For the literature, see [2, 3, 4].

Theorem 1.1 ([1]). *If ζ_1 and ζ_2 are non-negative real-valued convex functions on $[\mathfrak{z}_1, \mathfrak{z}_2]$, then*

$$\frac{1}{\mathfrak{z}_2 - \mathfrak{z}_1}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \zeta_1(\varphi)\zeta_2(\varphi)\,d\varphi \leq \frac{1}{3}M(\mathfrak{z}_1,\mathfrak{z}_2) + \frac{1}{6}N(\mathfrak{z}_1,\mathfrak{z}_2),$$

and

$$2\zeta_1\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\zeta_2\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right) \leq \frac{1}{\mathfrak{z}_2 - \mathfrak{z}_1}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \zeta_1(\varphi)\zeta_2(\varphi)\,d\varphi + \frac{1}{3}M(\mathfrak{z}_1,\mathfrak{z}_2) + \frac{1}{6}N(\mathfrak{z}_1,\mathfrak{z}_2),$$

where

$$\begin{aligned} M(u_1,\mathfrak{z}_2) &= \zeta_1(\mathfrak{z}_1)\zeta_2(\mathfrak{z}_1) + \zeta_1(\mathfrak{z}_2)\zeta_2(\mathfrak{z}_2), \\ N(u_1,\mathfrak{z}_2) &= \zeta_1(\mathfrak{z}_1)\zeta_2(\mathfrak{z}_2) + \zeta_1(\mathfrak{z}_2)\zeta_2(\mathfrak{z}_1). \end{aligned} \tag{1.1}$$

The structure of the article is organized as follows. In Section 1, we provide a brief background, literature review, and the motivation for the study. Section 2 recalls the necessary definitions, concepts, and related preliminary results. The main results of the paper are presented in Section 3. Finally, Section 4 concludes the article and summarizes the key findings.

2. Preliminaries

This section provides the foundational concepts and definitions necessary for our main results.

The concept of harmonic convexity has gained a prominent generalization of classical convexity in recent years and has given new insights into the fields of mathematical analysis and inequalities. The idea of harmonic convex functions was introduced in the seminal work by İscan [5] in the following form.

Definition 2.1 (İscan [5]). *A function $\zeta : \mathcal{K} \subset \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is called harmonically convex if*

$$\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}_1\mathfrak{z}_1 + (1-\mathfrak{d}_1)\mathfrak{z}_2}\right) \leq \mathfrak{d}_1\zeta(\mathfrak{z}_1) + (1-\mathfrak{d}_1)\zeta(\mathfrak{z}_2),$$

for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{K}$ and $\mathfrak{d}_1 \in [0,1]$.

Definition 2.2 (Harmonical h-Godunova–Levin functions [6]). *Let $h : (0,1) \subseteq \mathbf{R} \to \mathbf{R}$ be a non-negative function. A function $\zeta : \mathcal{K} \subset \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is said to be* harmonically h-Godunova–Levin *if*

$$\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}_1\mathfrak{z}_1 + (1-\mathfrak{d}_1)\mathfrak{z}_2}\right) \leq \frac{\zeta(\mathfrak{z}_1)}{h(\mathfrak{d}_1)} + \frac{\zeta(\mathfrak{z}_2)}{h(1-\mathfrak{d}_1)},$$

for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{K}$ and $\mathfrak{d}_1 \in (0,1)$.

3. Main Results

In this section, we establish several new variants of Pachpatte-type by utilizing the fractional conformable integral operator. Before presenting the main results, we briefly recall the fundamental definitions and properties of this operator, which will be used throughout the section.

3.1. Fractional Conformable Integral Operator

Let $\alpha > 0$, $\mathrm{Re}(\beta) > 0$ and $f \in C[\theta, \xi]$. The left- and right-sided fractional conformable integral operators are defined, respectively, by

$$^{\beta}J^{\alpha}_{\theta+} f(\varkappa) := \frac{1}{\Gamma(\beta)} \int_{\theta}^{\varkappa} \left(\frac{(\varkappa-\theta)^{\alpha} - (\tau-\theta)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(\tau)}{(\tau-\theta)^{1-\alpha}} d\tau, \quad \varkappa > \theta,$$

and

$$^{\beta}J^{\alpha}_{\xi-} f(\varkappa) := \frac{1}{\Gamma(\beta)} \int_{\varkappa}^{\xi} \left(\frac{(\xi-\varkappa)^{\alpha} - (\xi-\tau)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(\tau)}{(\xi-\tau)^{1-\alpha}} d\tau, \quad \varkappa < \xi.$$

Theorem 3.1. *Let $\zeta, \Upsilon : [\mathfrak{z}_1, \mathfrak{z}_2] \to \mathbb{R}$ be two harmonically h-Godunova–Levin functions defined on $[\mathfrak{z}_1, \mathfrak{z}_2]$ with $0 < \mathfrak{z}_1 < \mathfrak{z}_2$, and $\zeta, \Upsilon \in L^1[\mathfrak{z}_1, \mathfrak{z}_2]$. Let $h : (0,1) \to \mathbb{R}^+$ satisfy $h(\eth) \neq 0$ for all $\eth \in (0,1)$. Then, for $\alpha > 0$ and $\mathrm{Re}(\beta) > 0$:*

$$\begin{aligned}
&\frac{1}{2\Gamma(\beta)} \left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2 - \mathfrak{z}_1} \right)^{\alpha(\beta-1)+1} \left\{ \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}} \left(\frac{(\frac{1}{\mathfrak{z}_1} - \frac{1}{\mathfrak{z}_2})^{\alpha} - (t - \frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha} \right)^{\beta-1} \frac{((\zeta\Upsilon)\circ\chi)(t)}{(t - \frac{1}{\mathfrak{z}_2})^{1-\alpha}}\, dt \right. \\
&\quad \left. + \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}} \left(\frac{(\frac{1}{\mathfrak{z}_1} - \frac{1}{\mathfrak{z}_2})^{\alpha} - (\frac{1}{\mathfrak{z}_1} - t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{((\zeta\Upsilon)\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1} - t)^{1-\alpha}}\, dt \right\} \\
&\leq \frac{\zeta(\mathfrak{z}_1)\Upsilon(\mathfrak{z}_1) + \zeta(\mathfrak{z}_2)\Upsilon(\mathfrak{z}_2)}{2} \int_0^1 \frac{\eth^{\alpha(\beta-1)}}{h^2(\eth) + h^2(1-\eth)}\, d\eth \\
&\quad + [\zeta(\mathfrak{z}_1)\Upsilon(\mathfrak{z}_2) + \zeta(\mathfrak{z}_2)\Upsilon(\mathfrak{z}_1)] \int_0^1 \frac{\eth^{\alpha(\beta-1)}}{h(\eth)h(1-\eth)}\, d\eth,
\end{aligned} \tag{1}$$

where $\chi(t) = 1/t$ for $t > 0$.

Proof. By the harmonic h-Godunova–Levin property of ζ and Υ, for $\eth \in (0,1)$:

$$\zeta\left(\tfrac{\mathfrak{z}_1\mathfrak{z}_2}{\eth\mathfrak{z}_2 + (1-\eth)\mathfrak{z}_1} \right) \leq \frac{\zeta(\mathfrak{z}_1)}{h(\eth)} + \frac{\zeta(\mathfrak{z}_2)}{h(1-\eth)}, \tag{2}$$

$$\Upsilon\left(\tfrac{\mathfrak{z}_1\mathfrak{z}_2}{\eth\mathfrak{z}_2 + (1-\eth)\mathfrak{z}_1} \right) \leq \frac{\Upsilon(\mathfrak{z}_1)}{h(\eth)} + \frac{\Upsilon(\mathfrak{z}_2)}{h(1-\eth)}. \tag{3}$$

Multiplying (2)–(3) and repeating with the symmetric argument $\eth\mathfrak{z}_1 + (1-\eth)\mathfrak{z}_2$, then adding the two resulting inequalities gives:

$$\begin{aligned}
&(\zeta\Upsilon)\left(\tfrac{\mathfrak{z}_1\mathfrak{z}_2}{\eth\mathfrak{z}_2 + (1-\eth)\mathfrak{z}_1} \right) + (\zeta\Upsilon)\left(\tfrac{\mathfrak{z}_1\mathfrak{z}_2}{\eth\mathfrak{z}_1 + (1-\eth)\mathfrak{z}_2} \right) \\
&\leq \frac{\zeta(\mathfrak{z}_1)\Upsilon(\mathfrak{z}_1) + \zeta(\mathfrak{z}_2)\Upsilon(\mathfrak{z}_2)}{h^2(\eth) + h^2(1-\eth)} + \frac{2\big[\zeta(\mathfrak{z}_1)\Upsilon(\mathfrak{z}_2) + \zeta(\mathfrak{z}_2)\Upsilon(\mathfrak{z}_1)\big]}{h(\eth)h(1-\eth)}.
\end{aligned} \tag{4}$$

Multiplying (4) by $\mathfrak{d}^{\alpha(\beta-1)}$ and integrating over $(0,1)$ yields the left-hand side integrals $I_1 + I_2$. Applying the substitutions

$$t = \tfrac{1}{\mathfrak{z}_1} - \tfrac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathfrak{z}_1\mathfrak{z}_2}\mathfrak{d} \quad \text{and} \quad t = \tfrac{1}{\mathfrak{z}_2} + \tfrac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathfrak{z}_1\mathfrak{z}_2}\mathfrak{d}$$

respectively, with $\chi(t) = 1/t$, transforms these into the conformable fractional integrals:

$$I_1 + I_2 = \frac{1}{\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^\alpha-(t-\frac{1}{\mathfrak{z}_2})^\alpha}{\alpha}\right)^{\beta-1}\frac{((\zeta\Upsilon)\circ\chi)(t)}{(t-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,dt\right.$$
$$\left.+\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^\alpha-(\frac{1}{\mathfrak{z}_1}-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{((\zeta\Upsilon)\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\,dt\right\}.$$

Substituting into the integrated form of (4) and dividing by 2 yields (1). □

Example 3.2. *Under the assumptions of Theorem 3.1, let $\mathfrak{z}_1 = 1.2$, $\mathfrak{z}_2 = 3$, $\alpha = 1.5$, $\beta = 0.9$, with*

$$\zeta(\xi) = \xi^2, \qquad \Upsilon(\xi) = \sqrt{\xi}, \qquad h(\mathfrak{d}) = \mathfrak{d} + 0.7, \quad \mathfrak{d} \in (0,1),$$

and $\chi(\xi) = 1/\xi$, so that $(\zeta\Upsilon\circ\chi)(\xi) = \xi^{-5/2}$.

Left-hand side. *The harmonic mean is $H = \dfrac{2\cdot 1.2\cdot 3}{4.2} \approx 1.7143$, and $h(\frac{1}{2}) = 1.2$, so*

$$\text{LHS} = \frac{h(1/2)}{2}\zeta(H)\Upsilon(H) = \frac{1.2}{2}\cdot\left(1.7143^2\cdot\sqrt{1.7143}\right) = 0.6\times 3.847 \approx 2.308.$$

Middle term. *The key constants are*

$$\tfrac{1}{\mathfrak{z}_1} = 0.8\overline{3}, \quad \tfrac{1}{\mathfrak{z}_2} = 0.\overline{3}, \quad (0.5)^{1.5} \approx 0.3536, \quad \left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{0.85} = (2.0)^{0.85} \approx 1.822, \quad \Gamma(0.9) \approx 1.0686.$$

By symmetry of the two conformable fractional integrals, $I_1 = I_2 \approx 1.642$, giving

$$\text{Middle} = \frac{1.822}{2\times 1.0686}(1.642 + 1.642) = 0.8527\times 3.284 \approx 2.800.$$

Right-hand side. *The endpoint products are*

$$\zeta(\mathfrak{z}_1)\Upsilon(\mathfrak{z}_1) \approx 1.577, \quad \zeta(\mathfrak{z}_2)\Upsilon(\mathfrak{z}_2) \approx 15.589, \quad \zeta(\mathfrak{z}_1)\Upsilon(\mathfrak{z}_2) \approx 2.495, \quad \zeta(\mathfrak{z}_2)\Upsilon(\mathfrak{z}_1) \approx 9.859.$$

With $\alpha(\beta-1) = -0.15$ and $h(\mathfrak{d}) = \mathfrak{d} + 0.7$, numerical integration gives

$$J_1 = \int_0^1 \frac{\mathfrak{d}^{-0.15}}{(\mathfrak{d}+0.7)^2 + (1.7-\mathfrak{d})^2}\,d\mathfrak{d} \approx 0.924, \qquad J_2 = \int_0^1 \frac{\mathfrak{d}^{-0.15}}{(\mathfrak{d}+0.7)(1.7-\mathfrak{d})}\,d\mathfrak{d} \approx 1.178,$$

hence

$$\text{RHS} = \frac{1.577 + 15.589}{2}\times 0.924 + (2.495 + 9.859)\times 1.178 \approx 7.931 + 14.553 \approx 22.484.$$

The inequality chain

$$2.308 \;\approx\; \text{LHS} \;<\; \text{Middle} \;\approx\; 2.800 \;<\; \text{RHS} \;\approx\; 22.484$$

confirms Theorem 3.1 for the chosen parameters.

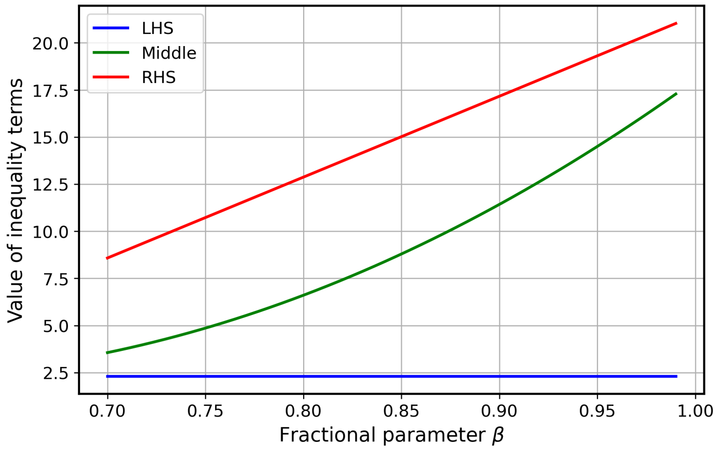

Figure 1: Graphical verification of the inequality $\mathrm{LHS} < \mathrm{Middle} < \mathrm{RHS}$ for Example 3.2.

Theorem 3.3. *Let $\zeta : [\mathfrak{z}_1, \mathfrak{z}_2] \to \mathbb{R}$ be a harmonical h-Godunova–Levin function, where $0 < \mathfrak{z}_1 < \mathfrak{z}_2$ and $\zeta \in L^1[\mathfrak{z}_1, \mathfrak{z}_2]$. Assume that the auxiliary function $h : (0,1) \to \mathbb{R}^+$ is positive and satisfies $h(\mathfrak{d}) \neq 0$ for all $\mathfrak{d} \in (0,1)$. Then, for every $\alpha > 0$ and $\mathrm{Re}(\beta) > 0$, the following double inequality holds:*

$$
\begin{aligned}
&\frac{h\left(\frac{1}{2}\right)}{2}\zeta\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right) \\
&\leq \frac{1}{2\Gamma(\beta)}\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{\int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2})^{\alpha}-(\frac{1}{\mathfrak{z}_1}-\tau)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(\tau)}{(\frac{1}{\mathfrak{z}_1}-\tau)^{1-\alpha}}\,d\tau\right. \\
&\qquad\left.+\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}\left(\frac{(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(\tau-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(\tau)}{(\tau-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,d\tau\right\} \\
&\leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h\left(\frac{2-\mathfrak{d}}{2}\right)}+\frac{1}{h\left(\frac{\mathfrak{d}}{2}\right)}\right]d\mathfrak{d}, \qquad (5)
\end{aligned}
$$

where $\chi(x) = 1/x$ for $x > 0$.

Proof. For $\mathfrak{d} \in (0,1)$, define:

$$c_1 = \frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(2-\mathfrak{d})\mathfrak{z}_2}, \quad e_1 = \frac{2\mathfrak{z}_1\mathfrak{z}_2}{(2-\mathfrak{d})\mathfrak{z}_1+\mathfrak{d}\mathfrak{z}_2}.$$

Then we compute:

$$\frac{2c_1e_1}{c_1+e_1} = \frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}.$$

From the harmonic h-Godunova–Levin property:

$$\zeta\left(\frac{2c_1e_1}{c_1+e_1}\right) \leq \frac{1}{h(1/2)}[\zeta(c_1)+\zeta(e_1)].$$

Thus:

$$h(1/2)\zeta\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right) \leq \zeta(c_1)+\zeta(e_1).$$

That is:

$$h(1/2)\zeta\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right) \le \zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(2-\mathfrak{d})\mathfrak{z}_2}\right)+\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(2-\mathfrak{d})\mathfrak{z}_1}\right). \tag{6}$$

Multiply both sides by $\mathfrak{d}^{\alpha(\beta-1)}$ and integrate over $(0,1)$:

$$\begin{aligned} &h(1/2)\zeta\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}d\mathfrak{d} \\ &\le \int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(2-\mathfrak{d})\mathfrak{z}_2}\right)d\mathfrak{d}+\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(2-\mathfrak{d})\mathfrak{z}_1}\right)d\mathfrak{d}. \end{aligned} \tag{7}$$

Note that $\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}d\mathfrak{d} = \frac{1}{\alpha(\beta-1)+1}$.
Now transform the integrals. For the first integral:

$$I_1 = \int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(2-\mathfrak{d})\mathfrak{z}_2}\right)d\mathfrak{d}.$$

Set $t=\frac{1}{\mathfrak{z}_1}-\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2\mathfrak{z}_1\mathfrak{z}_2}\mathfrak{d}$. Then:

$$\mathfrak{d}=\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\left(\frac{1}{\mathfrak{z}_1}-t\right), \quad d\mathfrak{d}=-\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}dt, \quad \frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(2-\mathfrak{d})\mathfrak{z}_2}=\frac{1}{t}.$$

When $\mathfrak{d}=0$, $t=\frac{1}{\mathfrak{z}_1}$; when $\mathfrak{d}=1$, $t=\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}$.
Thus:

$$\begin{aligned} I_1 &= \int_{1/\mathfrak{z}_1}^{(\mathfrak{z}_1+\mathfrak{z}_2)/(2\mathfrak{z}_1\mathfrak{z}_2)}\left[\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\left(\frac{1}{\mathfrak{z}_1}-t\right)\right]^{\alpha(\beta-1)}(\zeta\circ\chi)(t)\left(-\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)dt \\ &= \left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\int_{(\mathfrak{z}_1+\mathfrak{z}_2)/(2\mathfrak{z}_1\mathfrak{z}_2)}^{1/\mathfrak{z}_1}\left(\frac{1}{\mathfrak{z}_1}-t\right)^{\alpha(\beta-1)}(\zeta\circ\chi)(t)dt. \end{aligned}$$

Now rewrite in fractional conformable form:

$$I_1=\frac{1}{\Gamma(\beta)}\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2})^\alpha-(\frac{1}{\mathfrak{z}_1}-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\,dt.$$

For the second integral:

$$I_2 = \int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(2-\mathfrak{d})\mathfrak{z}_1}\right)d\mathfrak{d}.$$

Set $t=\frac{1}{\mathfrak{z}_2}+\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2\mathfrak{z}_1\mathfrak{z}_2}\mathfrak{d}$. Then:

$$\mathfrak{d}=\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\left(t-\frac{1}{\mathfrak{z}_2}\right), \quad d\mathfrak{d}=\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}dt, \quad \frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(2-\mathfrak{d})\mathfrak{z}_1}=\frac{1}{t}.$$

When $\mathfrak{d}=0$, $t=\frac{1}{\mathfrak{z}_2}$; when $\mathfrak{d}=1$, $t=\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}$.
Thus:

$$\begin{aligned} I_2 &= \int_{1/\mathfrak{z}_2}^{(\mathfrak{z}_1+\mathfrak{z}_2)/(2\mathfrak{z}_1\mathfrak{z}_2)}\left[\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\left(t-\frac{1}{\mathfrak{z}_2}\right)\right]^{\alpha(\beta-1)}(\zeta\circ\chi)(t)\cdot\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}dt \\ &= \left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\int_{1/\mathfrak{z}_2}^{(\mathfrak{z}_1+\mathfrak{z}_2)/(2\mathfrak{z}_1\mathfrak{z}_2)}\left(t-\frac{1}{\mathfrak{z}_2}\right)^{\alpha(\beta-1)}(\zeta\circ\chi)(t)dt. \end{aligned}$$

Rewrite in fractional conformable form:

$$I_2 = \frac{1}{\Gamma(\beta)} \left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2 - \mathfrak{z}_1} \right)^{\alpha(\beta-1)+1} \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}} \left(\frac{(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2} - \frac{1}{\mathfrak{z}_2})^\alpha - (t - \frac{1}{\mathfrak{z}_2})^\alpha}{\alpha} \right)^{\beta-1} \frac{(\zeta \circ \chi)(t)}{(t - \frac{1}{\mathfrak{z}_2})^{1-\alpha}}\, dt.$$

Substituting into (7):

$$\begin{aligned} & h(1/2)\zeta\left(\frac{\mathfrak{z}_1 + \mathfrak{z}_2}{2}\right) \frac{1}{\alpha(\beta-1)+1} \\ & \leq \frac{1}{\Gamma(\beta)} \left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2 - \mathfrak{z}_1} \right)^{\alpha(\beta-1)+1} \left\{ \int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}} \left(\frac{(\frac{1}{\mathfrak{z}_1} - \frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2})^\alpha - (\frac{1}{\mathfrak{z}_1} - t)^\alpha}{\alpha} \right)^{\beta-1} \frac{(\zeta \circ \chi)(t)}{(\frac{1}{\mathfrak{z}_1} - t)^{1-\alpha}}\, dt \right. \\ & \left. + \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}} \left(\frac{(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2} - \frac{1}{\mathfrak{z}_2})^\alpha - (t - \frac{1}{\mathfrak{z}_2})^\alpha}{\alpha} \right)^{\beta-1} \frac{(\zeta \circ \chi)(t)}{(t - \frac{1}{\mathfrak{z}_2})^{1-\alpha}}\, dt \right\}. \end{aligned}$$

Multiplying both sides by $\Gamma(\beta) \left(\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2\mathfrak{z}_1\mathfrak{z}_2} \right)^{\alpha(\beta-1)+1} [\alpha(\beta-1)+1]$ gives:

$$\begin{aligned} & h(1/2)\Gamma(\beta)[\alpha(\beta-1)+1] \left(\frac{\mathfrak{z}_2 - \mathfrak{z}_1}{2\mathfrak{z}_1\mathfrak{z}_2} \right)^{\alpha(\beta-1)+1} \zeta\left(\frac{\mathfrak{z}_1 + \mathfrak{z}_2}{2}\right) \\ & \leq \int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}} \left(\frac{(\frac{1}{\mathfrak{z}_1} - \frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2})^\alpha - (\frac{1}{\mathfrak{z}_1} - t)^\alpha}{\alpha} \right)^{\beta-1} \frac{(\zeta \circ \chi)(t)}{(\frac{1}{\mathfrak{z}_1} - t)^{1-\alpha}}\, dt \\ & + \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}} \left(\frac{(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2} - \frac{1}{\mathfrak{z}_2})^\alpha - (t - \frac{1}{\mathfrak{z}_2})^\alpha}{\alpha} \right)^{\beta-1} \frac{(\zeta \circ \chi)(t)}{(t - \frac{1}{\mathfrak{z}_2})^{1-\alpha}}\, dt. \end{aligned}$$

Dividing by $2\Gamma(\beta)$ and rearranging gives the first inequality in (5):

$$\begin{aligned} \frac{h(1/2)}{2}\zeta\left(\frac{\mathfrak{z}_1 + \mathfrak{z}_2}{2}\right) \leq \frac{1}{2\Gamma(\beta)} \left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2 - \mathfrak{z}_1} \right)^{\alpha(\beta-1)+1} & \left\{ \int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}} \left(\frac{(\frac{1}{\mathfrak{z}_1} - \frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2})^\alpha - (\frac{1}{\mathfrak{z}_1} - \tau)^\alpha}{\alpha} \right)^{\beta-1} \right. \\ \times \frac{(\zeta \circ \chi)(\tau)}{(\frac{1}{\mathfrak{z}_1} - \tau)^{1-\alpha}}\, d\tau + \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}} & \left. \left(\frac{(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2} - \frac{1}{\mathfrak{z}_2})^\alpha - (\tau - \frac{1}{\mathfrak{z}_2})^\alpha}{\alpha} \right)^{\beta-1} \frac{(\zeta \circ \chi)(\tau)}{(\tau - \frac{1}{\mathfrak{z}_2})^{1-\alpha}}\, d\tau \right\}. \end{aligned}$$

For the second inequality, we use the harmonic h-Godunova–Levin property directly:

$$\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1 + (2-\mathfrak{d})\mathfrak{z}_2}\right) \leq \frac{\zeta(\mathfrak{z}_1)}{h\left(\frac{2-\mathfrak{d}}{2}\right)} + \frac{\zeta(\mathfrak{z}_2)}{h\left(\frac{\mathfrak{d}}{2}\right)}, \tag{8}$$

$$\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2 + (2-\mathfrak{d})\mathfrak{z}_1}\right) \leq \frac{\zeta(\mathfrak{z}_1)}{h\left(\frac{2-\mathfrak{d}}{2}\right)} + \frac{\zeta(\mathfrak{z}_2)}{h\left(\frac{\mathfrak{d}}{2}\right)}. \tag{9}$$

Adding (8) and (9):

$$\zeta(c_1) + \zeta(e_1) \leq [\zeta(\mathfrak{z}_1) + \zeta(\mathfrak{z}_2)] \left[\frac{1}{h\left(\frac{2-\mathfrak{d}}{2}\right)} + \frac{1}{h\left(\frac{\mathfrak{d}}{2}\right)} \right]. \tag{10}$$

From (6) and (10):

$$h(1/2)\zeta\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right) \le \zeta(c_1)+\zeta(e_1) \le [\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)]\left[\frac{1}{h\left(\frac{2-\mathfrak{d}}{2}\right)}+\frac{1}{h\left(\frac{\mathfrak{d}}{2}\right)}\right].$$

Multiply by $\mathfrak{d}^{\alpha(\beta-1)}$ and integrate over $(0,1)$:

$$h(1/2)\zeta\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}d\mathfrak{d} \le [\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)]\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h\left(\frac{2-\mathfrak{d}}{2}\right)}+\frac{1}{h\left(\frac{\mathfrak{d}}{2}\right)}\right]d\mathfrak{d}.$$

Also from (10):

$$\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}[\zeta(c_1)+\zeta(e_1)]d\mathfrak{d} \le [\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)]\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h\left(\frac{2-\mathfrak{d}}{2}\right)}+\frac{1}{h\left(\frac{\mathfrak{d}}{2}\right)}\right]d\mathfrak{d}.$$

The left side is I_1+I_2, so substituting the expressions for I_1 and I_2:

$$\frac{1}{\Gamma(\beta)}\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{\int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2})^\alpha-(\frac{1}{\mathfrak{z}_1}-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\,dt\right.$$
$$\left.+\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}\left(\frac{(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}-\frac{1}{\mathfrak{z}_2})^\alpha-(t-\frac{1}{\mathfrak{z}_2})^\alpha}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(t-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,dt\right\}$$
$$\le [\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)]\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h\left(\frac{2-\mathfrak{d}}{2}\right)}+\frac{1}{h\left(\frac{\mathfrak{d}}{2}\right)}\right]d\mathfrak{d}.$$

Dividing both sides by $2\Gamma(\beta)$ gives the second inequality in (5):

$$\frac{1}{2\Gamma(\beta)}\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{\int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2})^\alpha-(\frac{1}{\mathfrak{z}_1}-\tau)^\alpha}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(\tau)}{(\frac{1}{\mathfrak{z}_1}-\tau)^{1-\alpha}}\,d\tau\right.$$
$$\left.+\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}}\left(\frac{(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2}-\frac{1}{\mathfrak{z}_2})^\alpha-(\tau-\frac{1}{\mathfrak{z}_2})^\alpha}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(\tau)}{(\tau-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,d\tau\right\}$$
$$\le \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h\left(\frac{2-\mathfrak{d}}{2}\right)}+\frac{1}{h\left(\frac{\mathfrak{d}}{2}\right)}\right]d\mathfrak{d}.$$

Combining both inequalities completes the proof of Theorem 3.3. □

Example 3.4. *Under the assumptions of Theorem 3.3, let*

$$\mathfrak{z}_1 = 1, \quad \mathfrak{z}_2 = 4, \quad \alpha = 1.2, \quad \beta = 0.85,$$

with

$$\zeta(\xi) = \frac{1}{\xi + 0.5}, \quad \xi \in [1,4], \qquad h(\mathfrak{d}) = 1 + \mathfrak{d}, \quad \mathfrak{d} \in (0,1).$$

Left-hand side. *Since* $\frac{\mathfrak{z}_1 + \mathfrak{z}_2}{2} = 2.5$ *and* $h\left(\frac{1}{2}\right) = 1.5$,

$$\text{LHS} = \frac{h(1/2)}{2}\,\zeta(2.5) = \frac{1.5}{2} \cdot \frac{1}{3} = 0.250.$$

Middle term. *Setting* $(\zeta \circ \chi)(\tau) = \frac{\tau}{1 + 0.5\tau}$ *with* $\chi(\xi) = 1/\xi$, *the key constants are*

$$\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2 - \mathfrak{z}_1} = \tfrac{8}{3}, \quad \alpha(\beta - 1) + 1 = 0.82, \quad \left(\tfrac{8}{3}\right)^{0.82} \approx 2.148, \quad \Gamma(0.85) \approx 1.109,$$

and the integration limits are $\frac{1}{\mathfrak{z}_1} = 1$, $\frac{1}{\mathfrak{z}_2} = 0.25$, $\frac{\mathfrak{z}_1 + \mathfrak{z}_2}{2\mathfrak{z}_1\mathfrak{z}_2} = 0.625$, *giving equal half-widths* $(0.375)^{1.2} \approx 0.298$. *Numerical evaluation of the two conformable fractional integrals yields* $I_1 \approx 0.328$ *and* $I_2 \approx 0.261$, *so*

$$\text{Middle} = \frac{2.148}{2 \times 1.109}(0.328 + 0.261) \approx 0.968 \times 0.589 \approx 0.571.$$

Right-hand side. *With* $\zeta(1) = \frac{2}{3}$ *and* $\zeta(4) = \frac{2}{9}$, *and using* $h\left(\frac{2-\mathfrak{d}}{2}\right) = \frac{6-\mathfrak{d}}{2}$, $h\left(\frac{\mathfrak{d}}{2}\right) = \frac{2+\mathfrak{d}}{2}$,

$$J = \int_0^1 \mathfrak{d}^{-0.18} \cdot 2\left(\frac{1}{6 - \mathfrak{d}} + \frac{1}{2 + \mathfrak{d}}\right) d\mathfrak{d} \approx 2.914,$$

hence

$$\text{RHS} = \frac{\zeta(\mathfrak{z}_1) + \zeta(\mathfrak{z}_2)}{2} \cdot J = \frac{0.6667 + 0.2222}{2} \times 2.914 \approx 1.295.$$

The inequality chain

$$0.250 \;=\; \text{LHS} \;<\; \text{Middle} \;\approx\; 0.571 \;<\; \text{RHS} \;\approx\; 1.295$$

confirms Theorem 3.3 for the chosen parameters.

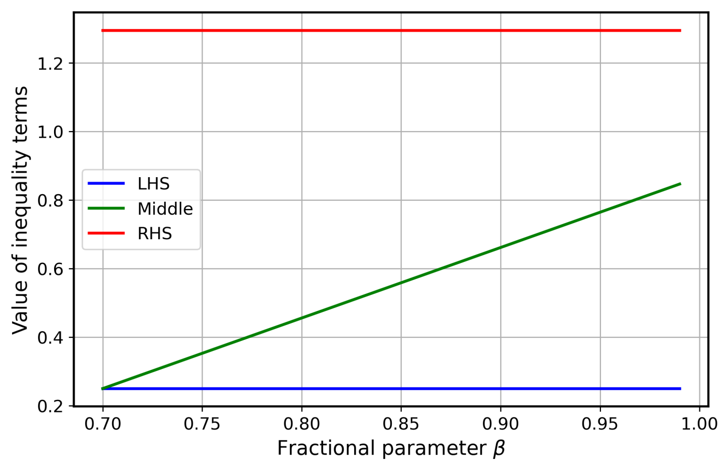

Figure 2: Graphical validation of Theorem 3.3 illustrating the ordering LHS $<$ Middle $<$ RHS .

Moreover, the curves of $\zeta(\xi)$ and $\zeta(\chi(\xi))$ are smooth, with $\zeta(\xi)$ decreasing and $\zeta(\chi(\xi))$ increasing on the considered interval, yielding well-behaved harmonic plots that visually support the analytical results.

4. Conclusion

We established the Pachpatte-type fractional inequalities involving harmonic h-Godunova–Levin convex functions in the frame of conformable fractional operator. The usefulness of these techniques is confirmed by numerical and graphical validations, especially for predicting quadrature errors over harmonic intervals.

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