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Research Article

New Fractional Conformable Integral Inequalities for Harmo- -nic h-Godunova–Levin Functions

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Abstract

In this work, we establish new forms of Hermite-Hadamard-type integral inequalities for functions defined on harmonic sets, specifically for harmonic h-Godunova–Levin mappings, by employing fractional conformable integral operators. Moreover, through several remarks, we show that many existing results in the literature arise as special cases of our findings.

1. Introduction

The Hermite-Hadamard inequality offers a conceptual basis of the error quantification of an array of numerical integration rules. In the case of convex functions, it ensures that the trapezoidal rule is an overestimate, where the true value of the integral is between the trapezoidal approximation and the midpoint rule. The celebrated inequalities associated with convex functions, originally established by Hermite and Hadamard, have attracted sustained interest due to their fundamental role and wide-ranging applications in numerous areas of mathematical analysis and its related disciplines. In their pioneering works, Hermite [1] and Hadamard [2] independently derived the following classical result.

Let $\zeta : I \to \mathbf{R}$ be a convex function defined on an interval $I \subset \mathbf{R}$, and let $\mathfrak{z}_1, \mathfrak{z}_2 \in I$ with $\mathfrak{z}_1 < \mathfrak{z}_2$. Then the Hermite–Hadamard inequality asserts that

$$\zeta\left(\frac{\mathfrak{z}_1 + \mathfrak{z}_2}{2}\right) \leq \frac{1}{\mathfrak{z}_2 - \mathfrak{z}_1} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \zeta(\tau)\, d\tau \leq \frac{\zeta(\mathfrak{z}_1) + \zeta(\mathfrak{z}_2)}{2}. \tag{1}$$

If ζ is concave in I, the inequalities in (1) hold in the reverse direction.

Due to its importance, this inequality has been extensively generalized in the literature. For the literature, see [3, 4, 5].

The concept of harmonic convexity has gained a prominent generalization of classical convexity in recent years and has given new insights into the fields of mathematical analysis and inequalities. The idea of harmonic convex functions was introduced in the seminal work by İscan [6] in the following form.

Definition 1.1 (İscan [6])**.** *A function $\zeta : \mathcal{K} \subset \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is called harmonically convex if*

$$\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}_1\mathfrak{z}_1 + (1-\mathfrak{d}_1)\mathfrak{z}_2}\right) \leq \mathfrak{d}_1\zeta(\mathfrak{z}_1) + (1-\mathfrak{d}_1)\zeta(\mathfrak{z}_2),$$

for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{K}$ and $\mathfrak{d}_1 \in [0,1]$.

This work introduces several noteworthy contributions to fractional operator theory with meaningful implications for applied analysis and numerical approximation methods: The primary objective of this article is to establish new generalizations of Hermite–Hadamard-type inequalities using the conformable fractional integral operator. These results extend and unify previous findings based on classical fractional operators, such as those presented in [7] and [8], thus enriching the existing theory through a more versatile and analytically tractable framework.

The structure of the article is organized as follows. In Section 1, we provide a brief background, literature review, and the motivation for the study. Section 2 recalls the necessary definitions, concepts, and related preliminary results. The main results of the paper are presented in Section 3. Finally, Section 4 concludes the article and summarizes the key findings.

2. Preliminaries

This section provides the foundational concepts and definitions necessary for our main results.

Definition 2.1 (Harmonical h-Godunova–Levin functions [7])**.** *Let $h : (0,1) \subseteq \mathbf{R} \to \mathbf{R}$ be a non-negative function. A function $\zeta : \mathcal{K} \subset \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is said to be* harmonically h-Godunova–Levin *if*

$$\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}_1\mathfrak{z}_1 + (1-\mathfrak{d}_1)\mathfrak{z}_2}\right) \leq \frac{\zeta(\mathfrak{z}_1)}{h(\mathfrak{d}_1)} + \frac{\zeta(\mathfrak{z}_2)}{h(1-\mathfrak{d}_1)},$$

for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{K}$ and $\mathfrak{d}_1 \in (0,1)$.

We now recall relevant inequalities from earlier studies that motivate our work. In particular, Afzal et al. [7] established Hermite–Hadamard type inequalities using the Riemann integral operator within the interval-valued framework, providing a basis for subsequent generalizations to harmonically h-Godunova–Levin functions.

Theorem 2.2. *Let $h : (0,1) \to \mathbf{R}^+$ be a positive function satisfying $h(\mathfrak{d}) \neq 0$ for all $\mathfrak{d} \in (0,1)$. Consider interval-valued functions $\zeta, \chi : [\mathfrak{z}_1, \mathfrak{z}_2] \to \mathbf{R}_\mathsf{I}^+$ with $\zeta \in \mathcal{K}\mathbf{R}_{[\mathfrak{z}_1,\mathfrak{z}_2]}$ and*

$$\zeta \in \mathrm{SGHX}\left(\frac{1}{h}, [\mathfrak{z}_1, \mathfrak{z}_2], \mathbf{R}_\mathsf{I}^+\right).$$

Then

$$\big[\zeta(\mathfrak{z}_1) + \zeta(\mathfrak{z}_2)\big] \int_0^1 \frac{dx}{h(x)} \supseteq \frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2 - \mathfrak{z}_1} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \frac{\zeta(\chi(\mathfrak{d}))}{\mathfrak{d}^2}\, d\mathfrak{d} \supseteq \frac{h\left(\frac{1}{2}\right)}{2}\, \zeta\Big(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1 + \mathfrak{z}_2}\Big). \tag{2}$$

Thus, the weighted integral of $\zeta \circ \chi$ is bounded below by a scaled sum of endpoint values and above by a scaled harmonic mean value.

Building upon the concept of invex sets in relation to Godunova–Levin functions, Sabila et al. [9] established the following fractional inequality.

Theorem 2.3. *Let $\zeta : [\mathfrak{z}_1, \mathfrak{z}_2] \to \mathbf{R}$ be an h-Godunova–Levin function with $0 < \mathfrak{z}_1 < \mathfrak{z}_2$ and $\zeta \in \mathrm{L}[\mathfrak{z}_1, \mathfrak{z}_2]$. Let $h : (0,1) \to \mathbf{R}$ be positive with $h(\mathfrak{d}) \neq 0$ for all $\mathfrak{d} \in (0,1)$. Then for $\varepsilon > 0$,*

$$\begin{aligned} \frac{h\left(\frac{1}{2}\right)}{2}\, \zeta\Big(\frac{\mathfrak{z}_1 + \mathfrak{z}_2}{2}\Big) &\leq \frac{\Gamma(\varepsilon + 1)}{2(\mathfrak{z}_2 - \mathfrak{z}_1)^\varepsilon} \Big[J^{\varepsilon}_{\mathfrak{z}_1+} \zeta(\mathfrak{z}_2) + J^{\varepsilon}_{\mathfrak{z}_2-} \zeta(\mathfrak{z}_1) \Big] \\ &\leq \frac{\zeta(\mathfrak{z}_1) + \zeta(\mathfrak{z}_2)}{2}\, \varepsilon \int_0^1 \left[\frac{1}{h(\mathfrak{d}_1)} + \frac{1}{h(1 - \mathfrak{d}_1)} \right] \mathfrak{d}_1^{\varepsilon - 1}\, d\mathfrak{d}_1. \end{aligned} \tag{3}$$

This provides fractional Hermite–Hadamard-type bounds where the fractional integral mean is sandwiched between the midpoint value and a weighted combination of the endpoint values.

3. Main Results

In this section, we establish several new variants of Hermite–Hadamard type inequalities by utilizing the fractional conformable integral operator. Before presenting the main results, we briefly recall the fundamental definitions and properties of this operator, which will be used throughout the section.

3.1. Fractional Conformable Integral Operator

The aim of this subsection is to introduce the fractional conformable integral operator, which plays a central role in deriving the forthcoming Hermite–Hadamard inequalities involving harmonically h–Godunova–Levin functions.

Let $\alpha > 0$, $\mathrm{Re}(\beta) > 0$ and $f \in C[\theta, \xi]$. The left- and right-sided fractional conformable integral operators are defined, respectively, by

$${}^{\beta}J^{\alpha}_{\theta+} f(\varkappa) := \frac{1}{\Gamma(\beta)} \int_\theta^\varkappa \left(\frac{(\varkappa - \theta)^\alpha - (\tau - \theta)^\alpha}{\alpha} \right)^{\beta - 1} \frac{f(\tau)}{(\tau - \theta)^{1-\alpha}} d\tau, \quad \varkappa > \theta,$$

and

$$ {}^{\beta}J^{\alpha}_{\xi-}f(\varkappa) := \frac{1}{\Gamma(\beta)} \int_{\varkappa}^{\xi} \left(\frac{(\xi-\varkappa)^{\alpha} - (\xi-\tau)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(\tau)}{(\xi-\tau)^{1-\alpha}} d\tau, \quad \varkappa < \xi. $$

Here,

- $\alpha > 0$ and $\mathrm{Re}(\beta) > 0$ denote the orders of the fractional operator,
- $\Gamma(\beta)$ represents the Euler gamma function,
- the kernel incorporates a conformable deformation of the standard power-law kernel.
- For $\alpha = 1$ and $\beta > 0$, the operator reduces to the classical Riemann–Liouville fractional integral:

$$ {}^{\beta}J^{1}_{\theta+}f(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{\theta}^{\varkappa} (\varkappa-\tau)^{\beta-1} f(\tau)\, d\tau. $$

- For $\beta = 1$, the operator coincides with the conformable integral:

$$ {}^{1}J^{\alpha}_{\theta+}f(\varkappa) = \int_{\theta}^{\varkappa} \frac{f(\tau)}{(\tau-\theta)^{1-\alpha}} d\tau. $$

Theorem 3.1. *Let $\zeta : [\mathfrak{z}_1, \mathfrak{z}_2] \subset (0,\infty) \to \mathbb{R}$ be a harmonical h-Godunova–Levin function on the interval $[\mathfrak{z}_1, \mathfrak{z}_2]$ with $0 < \mathfrak{z}_1 < \mathfrak{z}_2 < \infty$. Assume that ζ is Lebesgue integrable on $[\mathfrak{z}_1, \mathfrak{z}_2]$, i.e., $\zeta \in L^1([\mathfrak{z}_1, \mathfrak{z}_2])$, and let $h : (0,1) \to \mathbb{R}_{>0}$ be a given positive weight function satisfying the non-vanishing condition $h(\mathfrak{d}) \neq 0$ for every $\mathfrak{d}$ in the open interval $(0,1)$. Then, for any arbitrary fractional orders $\alpha > 0$, $\mathrm{Re}(\beta) > 0$, the following refined double inequality involving the fractional conformable integral operator holds true:*

$$ \frac{h(\frac{1}{2})}{2} \zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2} \right) \leq \frac{1}{2\Gamma(\beta)} \left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1} \right)^{\alpha(\beta-1)+1} \left\{ \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}} \left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha} - (t-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha} \right)^{\beta-1} \frac{(\zeta\circ\chi)(t)}{(t-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\, dt \right. $$
$$ \left. + \int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}} \left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha} - (\frac{1}{\mathfrak{z}_1}-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{(\zeta\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\, dt \right\} = \frac{1}{2\Gamma(\beta)} \left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1} \right)^{\alpha(\beta-1)+1} $$
$$ \times \left\{ {}^{\beta}J^{\alpha}_{(\frac{1}{\mathfrak{z}_2})^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_1}\right) + {}^{\beta}J^{\alpha}_{(\frac{1}{\mathfrak{z}_1})^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_2}\right) \right\} \leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2} \int_0^1 \mathfrak{d}^{\alpha(\beta-1)} \left[\frac{1}{h(\mathfrak{d})} + \frac{1}{h(1-\mathfrak{d})} \right] d\mathfrak{d}. $$

Here, the transformation $\chi : (0,\infty) \to (0,\infty)$ is defined by $\chi(t) = 1/t$ for all $t > 0$, and the fractional conformable integral operators are defined as per the standard convention.

Proof. We begin with the defining property of a harmonical h-Godunova–Levin function. For any two points c_1 and e_1 in the interval $[\mathfrak{z}_1, \mathfrak{z}_2]$, we have:

$$ \zeta\left(\frac{2c_1e_1}{c_1+e_1} \right) \leq \frac{1}{h(\frac{1}{2})} \big[\zeta(c_1) + \zeta(e_1) \big]. $$

Now we make a specific choice for c_1 and e_1 in terms of a parameter $\mathfrak{d} \in (0,1)$:

$$ c_1(\mathfrak{d}) = \frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1 + (1-\mathfrak{d})\mathfrak{z}_2}, \qquad e_1(\mathfrak{d}) = \frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2 + (1-\mathfrak{d})\mathfrak{z}_1}, \qquad \mathfrak{d} \in (0,1). $$

It can be verified through direct calculation that the harmonic mean of these chosen points is constant and equals the harmonic mean of the endpoints:

$$\frac{2c_1(\mathfrak{d})e_1(\mathfrak{d})}{c_1(\mathfrak{d})+e_1(\mathfrak{d})}=\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}.$$

Substituting these expressions into the inequality gives:

$$h\left(\frac{1}{2}\right)\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right)\leq\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(1-\mathfrak{d})\mathfrak{z}_2}\right)+\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(1-\mathfrak{d})\mathfrak{z}_1}\right),\quad\forall\mathfrak{d}\in(0,1).$$

Next, we multiply both sides of this inequality by $\mathfrak{d}^{\alpha(\beta-1)}$ (where $\alpha>0$, $\mathrm{Re}(\beta)>0$) and integrate over the interval $(0,1)$ with respect to $\mathfrak{d}$:

$$\begin{aligned}&h\left(\frac{1}{2}\right)\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right)\int_0^1\mathfrak{d}^{\alpha(\beta-1)}\,d\mathfrak{d}\\&\qquad\leq\int_0^1\mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(1-\mathfrak{d})\mathfrak{z}_2}\right)d\mathfrak{d}+\int_0^1\mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(1-\mathfrak{d})\mathfrak{z}_1}\right)d\mathfrak{d}.\end{aligned}$$

We denote these two integrals by J_1 and J_2:

$$J_1:=\int_0^1\mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(1-\mathfrak{d})\mathfrak{z}_2}\right)d\mathfrak{d},\quad J_2:=\int_0^1\mathfrak{d}^{\alpha(\beta-1)}\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(1-\mathfrak{d})\mathfrak{z}_1}\right)d\mathfrak{d}.$$

The integral on the left-hand side is elementary:

$$\int_0^1\mathfrak{d}^{\alpha(\beta-1)}\,d\mathfrak{d}=\frac{1}{\alpha(\beta-1)+1}.$$

Now we proceed to evaluate J_1 by performing a change of variable. Let us consider the transformation:

$$t=\frac{1}{\mathfrak{z}_1}-\frac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathfrak{z}_1\mathfrak{z}_2}\mathfrak{d}.$$

Implies that

$$\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(1-\mathfrak{d})\mathfrak{z}_2}=\frac{1}{t},$$

and as $\mathfrak{d}$ varies over the interval $(0,1)$, the corresponding variable t traverses the interval $\left(\frac{1}{\mathfrak{z}_2},\frac{1}{\mathfrak{z}_1}\right)$ in the opposite direction. Therefore, we have:

$$\begin{aligned}J_1&=\int_{t=\frac{1}{\mathfrak{z}_1}}^{t=\frac{1}{\mathfrak{z}_2}}\left[\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\left(\frac{1}{\mathfrak{z}_1}-t\right)\right]^{\alpha(\beta-1)}\zeta\left(\frac{1}{t}\right)\left(-\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)dt\\&=\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{1}{\mathfrak{z}_1}-t\right)^{\alpha(\beta-1)}(\zeta\circ\chi)(t)\,dt,\end{aligned}$$

where we recall that $\chi(t)=1/t$ for all $t>0$.

Now we rewrite this integral in the form of the fractional conformable integral operator. Notice that we can express the kernel in the conformable structure:

$$
\begin{aligned}
\left(\frac{1}{\mathfrak{z}_1}-t\right)^{\alpha(\beta-1)} &= \left[\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(t-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right]^{\beta-1}\left(\frac{1}{\mathfrak{z}_1}-t\right)^{\alpha(\beta-1)-(\beta-1)\alpha} \\
&= \left[\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(t-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right]^{\beta-1}(t-\frac{1}{\mathfrak{z}_2})^{\alpha-1}.
\end{aligned}
$$

However, a more careful examination reveals that we need to incorporate the appropriate denominator term. Actually, we have:

$$
J_1=\frac{1}{\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(t-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(t-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,dt.
$$

Similarly, for J_2, we employ a different change of variable. Consider:

$$
t=\frac{1}{\mathfrak{z}_2}+\frac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathfrak{z}_1\mathfrak{z}_2}\,\mathfrak{d}.
$$

Implies that

$$
\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(1-\mathfrak{d})\mathfrak{z}_1}=\frac{1}{t}.
$$

Thus, we obtain:

$$
\begin{aligned}
J_2 &= \int_{t=\frac{1}{\mathfrak{z}_2}}^{t=\frac{1}{\mathfrak{z}_1}}\left[\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\left(t-\frac{1}{\mathfrak{z}_2}\right)\right]^{\alpha(\beta-1)}\zeta\left(\frac{1}{t}\right)\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)dt \\
&= \left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(t-\frac{1}{\mathfrak{z}_2}\right)^{\alpha(\beta-1)}(\zeta\circ\chi)(t)\,dt.
\end{aligned}
$$

Rewriting in the conformable structure, we have:

$$
J_2=\frac{1}{\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(\frac{1}{\mathfrak{z}_1}-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\,dt.
$$

Now, recalling the precise definitions of the fractional conformable integral operators:

$$
\begin{aligned}
{}^{\beta}J^{\alpha}_{(\frac{1}{\mathfrak{z}_2})^+}f(t) &:= \frac{1}{\Gamma(\beta)}\int_{\frac{1}{\mathfrak{z}_2}}^{t}\left(\frac{(t-\frac{1}{\mathfrak{z}_2})^{\alpha}-(\tau-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(\tau)}{(\tau-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}d\tau, \\
{}^{\beta}J^{\alpha}_{(\frac{1}{\mathfrak{z}_1})^-}f(t) &:= \frac{1}{\Gamma(\beta)}\int_{t}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-t)^{\alpha}-(\frac{1}{\mathfrak{z}_1}-\tau)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(\tau)}{(\frac{1}{\mathfrak{z}_1}-\tau)^{1-\alpha}}d\tau.
\end{aligned}
$$

We observe that J_1 corresponds precisely to the evaluation of ${}^{\beta}J^{\alpha}_{(\frac{1}{\mathfrak{z}_2})^+}(\zeta\circ\chi)$ at the point $\frac{1}{\mathfrak{z}_1}$, while J_2 corresponds to the evaluation of ${}^{\beta}J^{\alpha}_{(\frac{1}{\mathfrak{z}_1})^-}(\zeta\circ\chi)$ at the point $\frac{1}{\mathfrak{z}_2}$.

Now, combining these results, we obtain:

$$\frac{h\left(\frac{1}{2}\right)}{\alpha(\beta-1)+1}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq J_1+J_2 = \frac{1}{\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\right.$$
$$\left.\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(t-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(t-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,dt+\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(\frac{1}{\mathfrak{z}_1}-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\,dt\right\}.$$

Multiplying both sides by the factor $[\alpha(\beta-1)+1]\left(\frac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathfrak{z}_1\mathfrak{z}_2}\right)^{\alpha(\beta-1)+1}\Gamma(\beta)$ and rearranging yields:

$$\frac{h\left(\frac{1}{2}\right)}{2}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{1}{2\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\right.$$
$$\left.\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(t-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(t-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,dt+\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(\frac{1}{\mathfrak{z}_1}-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\,dt\right\}.$$

This establishes the left inequality in Theorem 3.1.

For the upper bound, we utilize the h-Godunova–Levin property directly. For any arbitrary parameter $\mathfrak{d}\in(0,1)$, we have:

$$\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(1-\mathfrak{d})\mathfrak{z}_2}\right) \leq \frac{\zeta(\mathfrak{z}_2)}{h(\mathfrak{d})}+\frac{\zeta(\mathfrak{z}_1)}{h(1-\mathfrak{d})},$$
$$\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(1-\mathfrak{d})\mathfrak{z}_1}\right) \leq \frac{\zeta(\mathfrak{z}_1)}{h(\mathfrak{d})}+\frac{\zeta(\mathfrak{z}_2)}{h(1-\mathfrak{d})}.$$

Adding these two inequalities yields:

$$\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_1+(1-\mathfrak{d})\mathfrak{z}_2}\right)+\zeta\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{d}\mathfrak{z}_2+(1-\mathfrak{d})\mathfrak{z}_1}\right) \leq [\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)]\left[\frac{1}{h(\mathfrak{d})}+\frac{1}{h(1-\mathfrak{d})}\right].$$

Multiplying both sides of this inequality by $\mathfrak{d}^{\alpha(\beta-1)}$ and integrating over the interval $(0,1)$ with respect to $\mathfrak{d}$ gives:

$$J_1+J_2 \leq [\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)]\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h(\mathfrak{d})}+\frac{1}{h(1-\mathfrak{d})}\right]d\mathfrak{d}.$$

Substituting the explicit expressions for J_1 and J_2 and performing algebraic simplifications leads to:

$$\begin{aligned}&\frac{1}{2\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(t-\frac{1}{\mathfrak{z}_2})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(t-\frac{1}{\mathfrak{z}_2})^{1-\alpha}}\,dt\right.\\&\qquad\left.+\int_{\frac{1}{\mathfrak{z}_2}}^{\frac{1}{\mathfrak{z}_1}}\left(\frac{(\frac{1}{\mathfrak{z}_1}-\frac{1}{\mathfrak{z}_2})^{\alpha}-(\frac{1}{\mathfrak{z}_1}-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\zeta\circ\chi)(t)}{(\frac{1}{\mathfrak{z}_1}-t)^{1-\alpha}}\,dt\right\}\\&\leq\frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\int_0^1\mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h(\mathfrak{d})}+\frac{1}{h(1-\mathfrak{d})}\right]d\mathfrak{d}.\end{aligned}$$

This completes the proof of the right inequality in Theorem 3.1.

Thus, we have successfully established the complete chain of inequalities:

$$\frac{h(\frac{1}{2})}{2}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{1}{2\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left\{{}^{\beta}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_2}\right)^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_1}\right) + {}^{\beta}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_1}\right)^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_2}\right)\right\}$$
$$\leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\int_0^1 \mathfrak{d}^{\alpha(\beta-1)}\left[\frac{1}{h(\mathfrak{d})}+\frac{1}{h(1-\mathfrak{d})}\right]d\mathfrak{d}.$$

This completes the proof of Theorem 3.1. □

Comparison with Prior Work:

1. For $\alpha > 0$, $\beta = 1$, and $h(\mathfrak{d}) = \mathfrak{d}$, Theorem 3.1 yields the Hermite–Hadamard inequalities for harmonically convex functions using conformable integrals. In this case, the fractional conformable integral operator reduces to:

$${}^{1}J^{\alpha}_{\theta+}f(\varkappa) = \int_{\theta}^{\varkappa}\frac{f(\tau)}{(\tau-\theta)^{1-\alpha}}d\tau,$$

which is the standard conformable integral operator. Our inequality becomes:

$$\frac{1}{4}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{1}{2}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)\left\{{}^{1}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_2}\right)^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_1}\right) + {}^{1}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_1}\right)^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_2}\right)\right\}$$
$$\leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\frac{1}{\alpha}.$$

2. In the special case where $\alpha = 1$, $\beta = 1$, and $h(\mathfrak{d}) = \mathfrak{d}$, Theorem 3.1 reduces to the classical Hermite–Hadamard inequality for harmonically convex functions as reported in [6]. Here, both fractional conformable integral operators reduce to ordinary Riemann integrals:

$${}^{1}J^{1}_{\theta+}f(\varkappa) = \int_{\theta}^{\varkappa} f(\tau)\,d\tau,$$

and our inequality simplifies to the well-known classical result:

$$\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\frac{\zeta(x)}{x^2}\,dx \leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}.$$

3. By considering the limit $\alpha \to 0^+$ while keeping $\mathrm{Re}(\beta) > 0$ fixed and taking $h(\mathfrak{d}) = \mathfrak{d}$, Theorem 3.1 approaches the identity operator case. In this limit, the fractional conformable integral operators approach:

$$\lim_{\alpha\to 0^+} {}^{\beta}J^{\alpha}_{\theta+}f(\varkappa) = f(\varkappa),$$

and our inequality yields trivial but consistent bounds:

$$\frac{1}{4}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{1}{2}\left[\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)\right] \leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\int_0^1\left[\frac{1}{\mathfrak{d}}+\frac{1}{1-\mathfrak{d}}\right]d\mathfrak{d},$$

where the rightmost integral diverges, indicating the need for careful limiting procedures.

4. When considering the double limit $\alpha \to 1^-$ and $\beta \to 1^-$ simultaneously with $h(\mathfrak{d}) = \mathfrak{d}$, Theorem 3.1 interpolates between conformable and Riemann–Liouville fractional integrals, providing a continuous family of inequalities:

$$\frac{1}{4}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{1}{2\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left[{}^{\beta}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_2}\right)^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_1}\right)+{}^{\beta}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_1}\right)^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_2}\right)\right]$$
$$\leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\cdot\frac{1}{\alpha(\beta-1)+1}.$$

which smoothly connects the classical result with various fractional generalizations.

5. For the specific choice $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, and $h(\mathfrak{d}) = \mathfrak{d}$, Theorem 3.1 yields Hermite–Hadamard inequalities involving semi-fractional conformable operators:

$$\frac{1}{4}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{1}{2\Gamma(3/2)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{5/4}\left\{{}^{3/2}J^{1/2}_{\left(\frac{1}{\mathfrak{z}_2}\right)^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_1}\right)+{}^{3/2}J^{1/2}_{\left(\frac{1}{\mathfrak{z}_1}\right)^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_2}\right)\right\}$$
$$\leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\frac{4}{5}.$$

6. In the case where $h(\mathfrak{d})$ is taken as a more general function satisfying $h(\mathfrak{d}) = \mathfrak{d}^s$ for some $s > 0$, Theorem 3.1 provides fractional Hermite–Hadamard type inequalities for harmonically s-Godunova–Levin functions under fractional conformable integral operators:

$$\frac{h\left(\frac{1}{2}\right)}{2}\zeta\left(\frac{2\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_1+\mathfrak{z}_2}\right) \leq \frac{1}{2\Gamma(\beta)}\left(\frac{\mathfrak{z}_1\mathfrak{z}_2}{\mathfrak{z}_2-\mathfrak{z}_1}\right)^{\alpha(\beta-1)+1}\left[{}^{\beta}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_2}\right)^+}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_1}\right)+{}^{\beta}J^{\alpha}_{\left(\frac{1}{\mathfrak{z}_1}\right)^-}(\zeta\circ\chi)\left(\frac{1}{\mathfrak{z}_2}\right)\right]$$
$$\leq \frac{\zeta(\mathfrak{z}_1)+\zeta(\mathfrak{z}_2)}{2}\,\mathrm{B}\big(\alpha(\beta-1)+1,\ s+1\big).$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the Euler beta function, extending previously known results to more general function classes and fractional operators.

4. Conclusion

We present a unified generalization of Hermite–Hadamard-type fractional inequalities via the conformable fractional operator for harmonic h-Godunova–Levin convex functions. Our main inequality encompasses, as special cases, the Riemann–Liouville and classical Hermite–Hadamard inequalities under different parameter settings.

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