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Research Article

New $(\mathfrak{p}, \mathfrak{q})$-Symmetric Hermite-Hadamard-Type Inequalities in Quantum Calculus

Muhammad Nasim Aftab[1], Hanaa Abu-Zinadah[2], Alina Alb Lupaş[3] and Cheng-Chi Lee[4,5,*]

[1]*Department of Mathematics, University of Engineering and Technology, Lahore 54890, Pakistan*
[2]*Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah, Saudi Arabia*
[3]*Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087, Oradea, Romania*
[4]*Department of Library and Information Science, Artificial Intelligence Development Center, Fu Jen Catholic University, New Taipei City, 24205, Taiwan*
[5]*Department of Computer Science and Information Engineering, Fintech and Blockchain Research Center, Asia University, Taichung City, 41354, Taiwan*

[*] Corresponding author: cclee@mail.fju.edu.tw

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Abstract

In this article, we first derive the basic properties of the left $(\mathfrak{p}, \mathfrak{q})$-symmetric difference operator. In the following, we establish a change of variables formula and the fundamental theorem of calculus for the left $(\mathfrak{p}, \mathfrak{q})$-symmetric definite integral operators. Moreover, we construct two new left $(\mathfrak{p}, \mathfrak{q})$-symmetric Hermite-Hadamard-type inequalities for convex differentiable functions and construct their generalized form. Lastly, the graphical illustration depicts the validity of the newly obtained results.

1. Introduction

An interesting variant of the classical calculus, quantum calculus—also known as $\mathfrak{q}$-calculus—works without the standard concept of limits. Mathematicians and physicists can examine functions in discrete sets and quantum systems using a difference operator in quantum calculus instead of the classical derivative [1, 2, 3]. Moreover, the application of quantum calculus is also applied in the field of statistics [4, 5, 6], numerical analysis [7], boundary value problems [8, 9], and inequality theory [10, 11, 12, 13]. In addition, the generalized version of quantum calculus is known as a post-quantum calculus [14, 15]. The application of this generalized calculus can also be seen in the field of integral inequalities: [16, 17, 18]. Furthermore, the symmetric version of quantum calculus is called symmetric quantum calculus [1, 19, 20]. The corresponding derivative and definite integral to the symmetric quantum calculus can be expressed as follows.

Definition 1.1. [1] Let $\mathcal{W} : [\mathfrak{s},\mathfrak{t}] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued mapping. Then the $\mathfrak{q}$-symmetric difference operator at $z \in [\mathfrak{s},\mathfrak{t}]$ can be stated as

$$\widetilde{\mathcal{D}}_{\mathfrak{q}}\mathcal{W}(z) = \frac{\mathcal{W}(\mathfrak{q}z) - \mathcal{W}(\mathfrak{q}^{-1}z)}{(\mathfrak{q}-\mathfrak{q}^{-1})z}, \quad z \neq 0.$$

Definition 1.2. [1] Let $\mathcal{W} : [\mathfrak{s},\mathfrak{t}] \longrightarrow \mathbb{R}$ be a continuous mapping. Then the definite $\mathfrak{q}$-symmetric integral over $[\mathfrak{s},\mathfrak{t}]$ is stated as

$$\int_{\mathfrak{s}}^{z} \mathcal{W}(u)\, \widetilde{\mathrm{d}}_{\mathfrak{q}}u = (1-\mathfrak{q}^2)z\sum_{n=0}^{\infty}\mathfrak{q}^{2n}\mathcal{W}\left(\mathfrak{q}^{2n+1}z\right), \quad for \;\; z \in [\mathfrak{s},\mathfrak{t}].$$

These definitions are extended by Nosheen et al. at the end-point $\mathfrak{s} \in [\mathfrak{s},\mathfrak{t}]$. These extended definitions of the derivative and definite integral can be expressed as follows.

Definition 1.3. [21] Let $\mathcal{W}$ be a continuous mapping. Then the ${}_{\mathfrak{s}}\mathfrak{q}$-symmetric difference operator at $z \in [\mathfrak{s},\mathfrak{t}]$ can be expressed as

$${}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{q}}\mathcal{W}(z) = \frac{\mathcal{W}(\mathfrak{q}z + (1-\mathfrak{q})\mathfrak{s}) - \mathcal{W}(\mathfrak{q}^{-1}z + (1-\mathfrak{q}^{-1})\mathfrak{s})}{(\mathfrak{q}-\mathfrak{q}^{-1})(z-\mathfrak{s})}, \quad z \neq \mathfrak{s}.$$

Definition 1.4. [21] Let $\mathcal{W} : [\mathfrak{s},\mathfrak{t}] \longrightarrow \mathbb{R}$ be a continuous mapping. Then the definite ${}_{\mathfrak{s}}\mathfrak{q}$-symmetric integral over $[\mathfrak{s},\mathfrak{t}]$ is stated as

$$\int_{\mathfrak{s}}^{z} \mathcal{W}(u)\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{q}}u = (1-\mathfrak{q}^2)(z-\mathfrak{s})\sum_{n=0}^{\infty}\mathfrak{q}^{2n}\mathcal{W}\left(\mathfrak{q}^{2n+1}z + \left(1-\mathfrak{q}^{2n+1}\right)\mathfrak{s}\right), \quad for \;\; z \in [\mathfrak{s},\mathfrak{t}].$$

Using these definitions, they derived several integral inequalities. However, Butt et al. constructed numerous Hermite-Hadamard-type inequalities for convex differentiable and coordinated convex differentiable functions [22, 23].

2. Material and Methods

Inspired by [1, 15, 16, 21], Aftab et al. recently presented a post-quantum version of symmetric quantum calculus called $(\mathfrak{p},\mathfrak{q})$-symmetric calculus [24]. The definitions of derivative and definite integral associated with $(\mathfrak{p},\mathfrak{q})$-symmetric calculus are mentioned below.

Definition 2.1. [24] Consider $\mathcal{W}$ is a continuous mapping, then the $(\mathfrak{p}, \mathfrak{q})$-symmetric difference operator at $z \in [\mathfrak{s}, \mathfrak{t}]$ can be stated as

$$\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(z) = \frac{\mathcal{W}(\mathfrak{p}\mathfrak{q}z) - \mathcal{W}(\mathfrak{p}^{-1}\mathfrak{q}^{-1}z)}{(\mathfrak{p}\mathfrak{q} - \mathfrak{p}^{-1}\mathfrak{q}^{-1})z}, \quad z \neq 0.$$

Definition 2.2. [24] Consider $\mathcal{W}$ is a continuous mapping, then the $(\mathfrak{p}, \mathfrak{q})$-symmetric definite integral over $[\mathfrak{s}, \mathfrak{t}]$ can be defined as

$$\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{W}(u)\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u = \int_{0}^{\mathfrak{t}} \mathcal{W}(u)\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u - \int_{0}^{\mathfrak{s}} \mathcal{W}(u)\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u,$$

here

$$\int_{0}^{z} \mathcal{W}(u)\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u = (1-\mathfrak{p}^2\mathfrak{q}^2)z\sum_{n=0}^{\infty}\mathfrak{p}^{2n}\mathfrak{q}^{2n}\mathcal{W}\left(\mathfrak{p}^{2n+1}\mathfrak{q}^{2n+1}z\right), \quad \text{for } z \in [\mathfrak{s}, \mathfrak{t}].$$

These definitions are extended by Butt et al. [25] at the left endpoint $\mathfrak{s} \in [\mathfrak{s}, \mathfrak{t}]$ and are given below.

Definition 2.3. [25] Consider $\mathcal{W}$ is a continuous mapping, then its left $(\mathfrak{p}, \mathfrak{q})$-symmetric difference operator at $z \in [\mathfrak{s}, \mathfrak{t}]$ is stated as

$$_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(z) = \frac{\mathcal{W}((1-\mathfrak{p}\mathfrak{q})\mathfrak{s}+\mathfrak{p}\mathfrak{q}z) - \mathcal{W}((1-\mathfrak{p}^{-1}\mathfrak{q}^{-1})\mathfrak{s}+\mathfrak{p}^{-1}\mathfrak{q}^{-1}z)}{(\mathfrak{p}\mathfrak{q}-\mathfrak{p}^{-1}\mathfrak{q}^{-1})(z-\mathfrak{s})}, \quad z \neq \mathfrak{s}. \tag{1}$$

Definition 2.4. [25] Consider $\mathcal{W}$ is a continuous mapping, then its left $(\mathfrak{p}, \mathfrak{q})$-symmetric definite integral on $[\mathfrak{s}, \mathfrak{t}]$ is defined as

$$\int_{\mathfrak{s}}^{z} \mathcal{W}(u)\,{}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u = (1-\mathfrak{p}^2\mathfrak{q}^2)(z-\mathfrak{s})\sum_{n=0}^{\infty}\mathfrak{p}^{2n}\mathfrak{q}^{2n}\mathcal{W}\left(\left(1-\mathfrak{p}^{2n+1}\mathfrak{q}^{2n+1}\right)\mathfrak{s}+\mathfrak{p}^{2n+1}\mathfrak{q}^{2n+1}z\right), \tag{2}$$

here $z \in [\mathfrak{s}, \mathfrak{t}]$.

Using (2), they studied several integral inequalities. Some of them are stated below.

Theorem 2.5. [25] If $\mathcal{W} : [\mathfrak{s}, \mathfrak{t}] \longrightarrow \mathbb{R}$ is convex differentiable function on $(\mathfrak{s}, \mathfrak{t})$, then

$$\mathcal{W}\left(\frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathfrak{s}+\mathfrak{p}\mathfrak{q}\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \leqslant \frac{\int_{\mathfrak{s}}^{\mathfrak{t}}\mathcal{W}(u)\,{}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u}{\mathfrak{t}-\mathfrak{s}} \leqslant \frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathcal{W}(\mathfrak{s})+\mathfrak{p}\mathfrak{q}\mathcal{W}(\mathfrak{t})}{1+\mathfrak{p}^2\mathfrak{q}^2} \tag{3}$$

is true for $0 < \mathfrak{p}, \mathfrak{q} < 1$.

Theorem 2.6. [25] If $\mathcal{W}$ is convex differentiable function on $(\mathfrak{s}, \mathfrak{t})$, then

$$\begin{aligned}&\mathcal{W}\left(\frac{\mathfrak{p}\mathfrak{q}\mathfrak{s}+(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) - \frac{(1-\mathfrak{p}\mathfrak{q})^2(\mathfrak{t}-\mathfrak{s})}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{p}\mathfrak{q}\mathfrak{s}+(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\\ &\leqslant \frac{\int_{\mathfrak{s}}^{\mathfrak{t}}\mathcal{W}(u)\,{}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u}{\mathfrak{t}-\mathfrak{s}} \leqslant \frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathcal{W}(\mathfrak{s})+\mathfrak{p}\mathfrak{q}\mathcal{W}(\mathfrak{t})}{1+\mathfrak{p}^2\mathfrak{q}^2}\end{aligned} \tag{4}$$

exists.

Theorem 2.7. [25] If $\mathcal{W}$ is convex differentiable function on $(\mathfrak{s}, \mathfrak{t})$, then

$$\begin{aligned}&\mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{t}}{2}\right) - \frac{(1-\mathfrak{p}\mathfrak{q})^2(\mathfrak{t}-\mathfrak{s})}{2(1+\mathfrak{p}^2\mathfrak{q}^2)}\mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{t}}{2}\right)\\ &\leqslant \frac{\int_{\mathfrak{s}}^{\mathfrak{t}}\mathcal{W}(u)\,{}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u}{\mathfrak{t}-\mathfrak{s}} \leqslant \frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathcal{W}(\mathfrak{s})+\mathfrak{p}\mathfrak{q}\mathcal{W}(\mathfrak{t})}{1+\mathfrak{p}^2\mathfrak{q}^2}\end{aligned} \tag{5}$$

exists.

3. Main Results

First, we construct some properties of the left $(\mathfrak{p},\mathfrak{q})$-symmetric derivative.

Theorem 3.1. *If $\mathcal{W}$ and $\mathcal{W}_1$ are the left $(\mathfrak{p},\mathfrak{q})$-symmetric differentiable mappings, then the following outcomes are true:*

1. *For any $\mathfrak{n}_1,\mathfrak{n}_2\in\mathbb{R}$, then ${}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{n}_1\mathcal{W}(z)+\mathfrak{n}_2\mathcal{W}_1(z))=\mathfrak{n}_1{}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}W(z)+\mathfrak{n}_2{}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}W_1(z)$.*

2. *Product rules:*

$$\begin{aligned}&{}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}(\mathcal{W}(z)\cdot\mathcal{W}_1(z))\\&\quad=\mathcal{W}_1(z\mathfrak{p}\mathfrak{q}+(1-\mathfrak{p}\mathfrak{q})\mathfrak{t}){}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(z)+\mathcal{W}(\mathfrak{p}^{-1}\mathfrak{q}^{-1}z+(1-\mathfrak{p}^{-1}\mathfrak{q}^{-1})\mathfrak{t}){}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}_1(z).\end{aligned}$$

Or

$$\begin{aligned}&{}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}(\mathcal{W}(z)\cdot\mathcal{W}_1(z))\\&\quad=\mathcal{W}(z\mathfrak{p}\mathfrak{q}+(1-\mathfrak{p}\mathfrak{q})\mathfrak{t}){}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}_1(z)+\mathcal{W}_1(\mathfrak{p}^{-1}\mathfrak{q}^{-1}z+(1-\mathfrak{p}^{-1}\mathfrak{q}^{-1})\mathfrak{t}){}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(z).\end{aligned}$$

3. *Quotient rule: For $\mathcal{W}_1(z\mathfrak{p}\mathfrak{q}+(1-\mathfrak{p}\mathfrak{q})\mathfrak{t})\mathcal{W}_1(\mathfrak{p}^{-1}\mathfrak{q}^{-1}z+(1-\mathfrak{p}^{-1}\mathfrak{q}^{-1})\mathfrak{t})\neq0$, then*

$${}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\left(\frac{\mathcal{W}(z)}{\mathcal{W}_1(z)}\right)=\frac{\mathcal{W}_1(z\mathfrak{p}\mathfrak{q}+(1-\mathfrak{p}\mathfrak{q})\mathfrak{t}){}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(z)-\mathcal{W}(z\mathfrak{p}\mathfrak{q}+(1-\mathfrak{p}\mathfrak{q})\mathfrak{t}){}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}_1(z)}{\mathcal{W}_1(z\mathfrak{p}\mathfrak{q}+(1-\mathfrak{p}\mathfrak{q})\mathfrak{t})\mathcal{W}_1(\mathfrak{p}^{-1}\mathfrak{q}^{-1}z+(1-\mathfrak{p}^{-1}\mathfrak{q}^{-1})\mathfrak{t})}.$$

Proof. Using (1), these can be proved easily. □

Some fundamental results corresponding to the left $(\mathfrak{p},\mathfrak{q})$-symmetric integrals are given below.

Theorem 3.2. *Consider $\mathcal{W}$ is a $(\mathfrak{p},\mathfrak{q})$-symmetric integrable mapping, then the following results hold:*

1. $\int_0^1\mathcal{W}(\mathfrak{s}(1-u)+u\mathfrak{t})\ {}_0\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u=\dfrac{1}{\mathfrak{t}-\mathfrak{s}}\int_{\mathfrak{s}}^{\mathfrak{t}}\mathcal{W}(u)\ {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u$

2. ${}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\int_{\mathfrak{s}}^{z}\mathcal{W}(u)\ {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u=\mathcal{W}(z)$

3. $\int_{\mathfrak{s}}^{z}{}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(u)\ {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u=\mathcal{W}(z)$

4. *For any $\mathfrak{s}_1\in(\mathfrak{s},\mathfrak{t})$, then* $\int_{\mathfrak{s}_1}^{\mathfrak{t}}{}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(u)\ {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u=\mathcal{W}(\mathfrak{t})-\mathcal{W}(\mathfrak{s}_1)$.

Proof. (1) Using (2)

$$\begin{aligned}\int_0^1\mathcal{W}(\mathfrak{s}(1-u)+u\mathfrak{t})\ {}_0\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u&=(1-\mathfrak{p}^2\mathfrak{q}^2)\sum_{n=0}^{\infty}\mathfrak{p}^{2n}\mathfrak{q}^{2n}\mathcal{W}\left(\left(1-\mathfrak{p}^{2n+1}\mathfrak{q}^{2n+1}\right)\mathfrak{s}+\mathfrak{p}^{2n+1}\mathfrak{q}^{2n+1}\mathfrak{t}\right)\\&=\frac{1}{\mathfrak{t}-\mathfrak{s}}\int_{\mathfrak{s}}^{\mathfrak{t}}\mathcal{W}(u)\ {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}u.\end{aligned}$$

(2) Let $\mathcal{W}_3(\boldsymbol{z}) = \int_{\mathfrak{s}}^{\boldsymbol{z}} \mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}$, using (1) and then (2), we have

$$\begin{aligned}
&{}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}} \int_{\mathfrak{s}}^{\boldsymbol{z}} \mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} = {}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}_3(\boldsymbol{z}) \\
&= \frac{\mathcal{W}_3((1-\mathfrak{p}^{-1}\mathfrak{q}^{-1})\mathfrak{s} + \mathfrak{p}^{-1}\mathfrak{q}^{-1}\boldsymbol{z}) - \mathcal{W}_3((1-\mathfrak{p}\mathfrak{q})\mathfrak{s} + \mathfrak{p}\mathfrak{q}\boldsymbol{z})}{(\mathfrak{p}^{-1}\mathfrak{q}^{-1} - \mathfrak{p}\mathfrak{q})(\boldsymbol{z}-\mathfrak{s})} \\
&= \sum_{n=0}^{\infty} \mathfrak{p}^{2n}\mathfrak{q}^{2n}\mathcal{W}\left(\left(1-\mathfrak{p}^{2n}\mathfrak{q}^{2n}\right)\mathfrak{s} + \mathfrak{p}^{2n}\mathfrak{q}^{2n}\boldsymbol{z}\right) - \sum_{n=1}^{\infty} \mathfrak{p}^{2n}\mathfrak{q}^{2n}\mathcal{W}\left(\left(1-\mathfrak{p}^{2n}\mathfrak{q}^{2n}\right)\mathfrak{s} + \mathfrak{p}^{2n}\mathfrak{q}^{2n}\boldsymbol{z}\right) \\
&= \mathcal{W}(\boldsymbol{z}).
\end{aligned}$$

(3) Reusing (2)

$$\begin{aligned}
&\int_{\mathfrak{s}}^{\boldsymbol{z}} {}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} \\
&= (1-\mathfrak{p}^2\mathfrak{q}^2)(\boldsymbol{z}-\mathfrak{s}) \sum_{n=0}^{\infty} \mathfrak{p}^{2n}\mathfrak{q}^{2n}\, {}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}\left(\left(1-\mathfrak{p}^{2n+1}\mathfrak{q}^{2n+1}\right)\mathfrak{s} + \mathfrak{p}^{2n+1}\mathfrak{q}^{2n+1}\boldsymbol{z}\right) \\
&= (1-\mathfrak{p}^2\mathfrak{q}^2)(\boldsymbol{z}-\mathfrak{s}) \sum_{n=0}^{\infty} \mathfrak{p}^{2n}\mathfrak{q}^{2n} \\
&\quad \times \frac{\mathcal{W}\left(\left(1-\mathfrak{p}^{2n}\mathfrak{q}^{2n}\right)\mathfrak{s} + \mathfrak{p}^{2n}\mathfrak{q}^{2n}\boldsymbol{z}\right) - \mathcal{W}\left(\left(1-\mathfrak{p}^{2n+2}\mathfrak{q}^{2n+2}\right)\mathfrak{s} + \mathfrak{p}^{2n+2}\mathfrak{q}^{2n+2}\boldsymbol{z}\right)}{(1-\mathfrak{p}^2\mathfrak{q}^2)(\boldsymbol{z}-\mathfrak{s})\mathfrak{p}^{2n}\mathfrak{q}^{2n}} \\
&= \mathcal{W}(\boldsymbol{z}).
\end{aligned}$$

(4) Let $\mathfrak{s}_1 \in (\mathfrak{s},\mathfrak{t})$, then using (3)

$$\int_{\mathfrak{s}_1}^{\mathfrak{t}} {}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} = \int_{\mathfrak{s}}^{\mathfrak{t}} {}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} - \int_{\mathfrak{s}}^{\mathfrak{s}_1} {}_{\mathfrak{s}}\widetilde{\mathcal{D}}_{\mathfrak{p},\mathfrak{q}}\mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} = \mathcal{W}(\mathfrak{t}) - \mathcal{W}(\mathfrak{s}_1).$$

□

Now, we establish some new Hermite-Hadamard-type inequalities in the $(\mathfrak{p},\mathfrak{q})$-symmetric calculus at $\mathfrak{s}$.

Theorem 3.3. *If $\mathcal{W}$ is a convex differentiable mapping on $(\mathfrak{s},\mathfrak{t})$, then*

$$\begin{aligned}
&\mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \frac{(\mathfrak{p}\mathfrak{q}-1)(\mathfrak{t}-\mathfrak{s})}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \\
&\leqslant \frac{\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}}{\mathfrak{t}-\mathfrak{s}} \leqslant \frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathcal{W}(\mathfrak{s}) + \mathfrak{p}\mathfrak{q}\mathcal{W}(\mathfrak{t})}{1+\mathfrak{p}^2\mathfrak{q}^2}
\end{aligned} \tag{6}$$

exists.

Proof. Knowing that the tangent line's equation at point $\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2} \in (\mathfrak{s},\mathfrak{t})$ can be obtained as

$$\mathcal{K}(\boldsymbol{u}) = \mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\boldsymbol{u} - \frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right).$$

Because $\mathcal{W}$ is convex on $[\mathfrak{s},\mathfrak{t}]$,

$$\mathcal{K}(\boldsymbol{u}) = \mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\boldsymbol{u} - \frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \leqslant \mathcal{W}(\boldsymbol{u})$$

Taking the left $\widetilde{(\mathfrak{p},\mathfrak{q})}$-Integral from $\mathfrak{s}$ to $\mathfrak{t}$,

$$\begin{aligned}
&\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{K}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} = \int_{\mathfrak{s}}^{\mathfrak{t}} \left[\mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\boldsymbol{u} - \frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\right] {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} \\
&= (\mathfrak{t}-\mathfrak{s})\mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\int_{\mathfrak{s}}^{\mathfrak{t}} \boldsymbol{u}\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} - (\mathfrak{t}-\mathfrak{s})\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \\
&= (\mathfrak{t}-\mathfrak{s})\left[\mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathfrak{s}+\mathfrak{p}\mathfrak{q}\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2} - \frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\right].
\end{aligned}$$

Thus,

$$\mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \frac{(\mathfrak{p}\mathfrak{q}-1)(\mathfrak{t}-\mathfrak{s})}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \leq \frac{\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}}{\mathfrak{t}-\mathfrak{s}}. \tag{7}$$

Combining (7) with the term on the right-hand side of (3), (6) is obtained. □

Remark 3.4. *If* $\mathfrak{p} \to 1^-$ *in Theorem 3.3, Theorem 3 of* [20] is obtained.

Theorem 3.5. *If* $\mathcal{W}$ *is a convex differentiable mapping on* $(\mathfrak{s},\mathfrak{t})$, *then*

$$\begin{aligned}
&\mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) - \frac{(1-\mathfrak{p}\mathfrak{q})^2(\mathfrak{t}-\mathfrak{s})}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \\
&\leqslant \frac{\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{W}(\mathrm{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\mathrm{u}}{\mathfrak{t}-\mathfrak{s}} \leqslant \frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathcal{W}(\mathfrak{s})+\mathfrak{p}\mathfrak{q}\mathcal{W}(\mathfrak{t})}{1+\mathfrak{p}^2\mathfrak{q}^2}
\end{aligned} \tag{8}$$

result exists.

Proof. Knowing that the equation of the tangent line at a point $\dfrac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2} \in (\mathfrak{s},\mathfrak{t})$ can be expressed as

$$\mathcal{K}_1(\boldsymbol{u}) = \mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\boldsymbol{u} - \frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right).$$

Because $\mathcal{W}$ is convex on $[\mathfrak{s},\mathfrak{t}]$, for all $\boldsymbol{u} \in [\mathfrak{s},\mathfrak{t}]$

$$\mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\boldsymbol{u} - \frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \leqslant \mathcal{W}(\boldsymbol{u})$$

holds.
Applying the left $\widetilde{(\mathfrak{p},\mathfrak{q})}$-Integral from $\mathfrak{s}$ to $\mathfrak{t}$,

$$\begin{aligned}
&\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} \geq \int_{\mathfrak{s}}^{\mathfrak{t}} \left[\mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\boldsymbol{u} - \frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\right] {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} \\
&= (\mathfrak{t}-\mathfrak{s})\mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\int_{\mathfrak{s}}^{\mathfrak{t}} \boldsymbol{u}\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} - (\mathfrak{t}-\mathfrak{s})\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \\
&= (\mathfrak{t}-\mathfrak{s})\left[\mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\left(\frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathfrak{s}+\mathfrak{p}\mathfrak{q}\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2} - \frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\right].
\end{aligned}$$

Consequently,

$$\mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \frac{\mathfrak{p}\mathfrak{q}(1-\mathfrak{p}\mathfrak{q})(\mathfrak{t}-\mathfrak{s})}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) \leqslant \frac{\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{W}(\boldsymbol{u})\, {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}}{\mathfrak{t}-\mathfrak{s}}. \tag{9}$$

Combining (9) with the right-hand side term of (3), the desired result has been obtained. □

Moreover, these results can be generalized as mentioned in the following.

Theorem 3.6. *If $\mathcal{W}$ is a convex differentiable on $(\mathfrak{s},\mathfrak{t})$, then*

$$\max\{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4,\mathcal{L}_5\} \leqslant \frac{\int_{\mathfrak{s}}^{\mathfrak{t}} \mathcal{W}(\boldsymbol{u})\ {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}}{\mathfrak{t}-\mathfrak{s}} \leqslant \frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathcal{W}(\mathfrak{s})+\mathfrak{p}\mathfrak{q}\mathcal{W}(\mathfrak{t})}{1+\mathfrak{p}^2\mathfrak{q}^2} \tag{10}$$

holds, here

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{W}\left(\frac{\mathfrak{s}(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)+\mathfrak{t}\mathfrak{p}\mathfrak{q}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right),\\
\mathcal{L}_2 &= \mathcal{W}\left(\frac{\mathfrak{s}\mathfrak{p}\mathfrak{q}+\mathfrak{t}(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) - \frac{(\mathfrak{t}-\mathfrak{s})(1-\mathfrak{p}\mathfrak{q})^2}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{s}\mathfrak{p}\mathfrak{q}+\mathfrak{t}(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)}{1+\mathfrak{p}^2\mathfrak{q}^2}\right),\\
\mathcal{L}_3 &= \mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{t}}{2}\right) - \frac{(\mathfrak{t}-\mathfrak{s})(1-\mathfrak{p}\mathfrak{q})^2}{2(1+\mathfrak{p}^2\mathfrak{q}^2)}\mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{t}}{2}\right),\\
\mathcal{L}_4 &= \mathcal{W}\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \frac{(\mathfrak{t}-\mathfrak{s})(\mathfrak{p}\mathfrak{q}-1)}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{p}^2\mathfrak{q}^2\mathfrak{s}+\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right),\\
\mathcal{L}_5 &= \mathcal{W}\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right) + \frac{\mathfrak{p}\mathfrak{q}(1-\mathfrak{p}\mathfrak{q})(\mathfrak{t}-\mathfrak{s})}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{\mathfrak{s}+\mathfrak{p}^2\mathfrak{q}^2\mathfrak{t}}{1+\mathfrak{p}^2\mathfrak{q}^2}\right).
\end{aligned}$$

Proof. Combining (3), (4), (5), (6), and (8), we get (10). □

3.1. Graphical Analysis

Now, we investigate our newly acquired results through graphs.

Example 3.7. *Suppose $\mathcal{W}(\boldsymbol{u})=\boldsymbol{u}^2$ in Theorem* 3.3.

Case 1: Put $\mathfrak{s}=0,\mathfrak{t}=1$, and the values of $\mathfrak{p},\mathfrak{q}\in\,]0,1[$, then (6) becomes

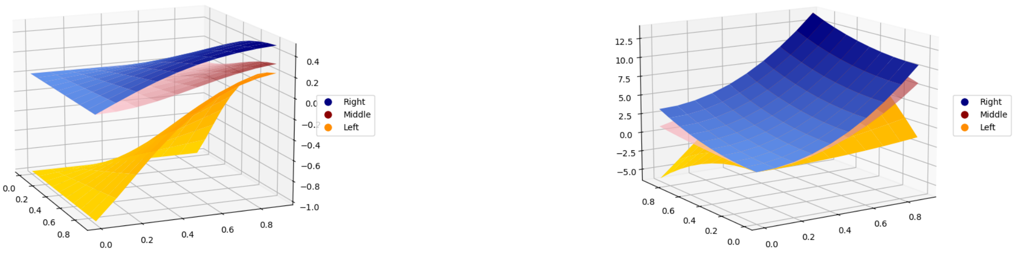
$$\begin{aligned}
&\mathcal{W}\left(\frac{1}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)+\frac{\mathfrak{p}\mathfrak{q}-1}{1+\mathfrak{p}^2\mathfrak{q}^2}\mathcal{W}'\left(\frac{1}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)\\
&\qquad\leqslant \int_0^1 \mathcal{W}(\boldsymbol{u})_0\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u} \leqslant \frac{(1-\mathfrak{p}\mathfrak{q}+\mathfrak{p}^2\mathfrak{q}^2)\mathcal{W}(0)+\mathfrak{p}\mathfrak{q}\mathcal{W}(1)}{1+\mathfrak{p}^2\mathfrak{q}^2}\\
&\left(\frac{1}{1+\mathfrak{p}^2\mathfrak{q}^2}\right)^2+\frac{2\mathfrak{p}\mathfrak{q}-2}{1+\mathfrak{p}^2\mathfrak{q}^2}\frac{1}{1+\mathfrak{p}^2\mathfrak{q}^2}\leqslant \int_0^1 \boldsymbol{u}^2{}_0\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}\leqslant \frac{1}{1+\mathfrak{p}^2\mathfrak{q}^2}(\mathfrak{p}\mathfrak{q})\\
&\frac{2\mathfrak{p}\mathfrak{q}-1}{1+2\mathfrak{p}^2\mathfrak{q}^2+\mathfrak{p}^4\mathfrak{q}^4}\leqslant\frac{1}{1+\mathfrak{p}^2\mathfrak{q}^2+\mathfrak{p}^4\mathfrak{q}^4}(\mathfrak{p}^2\mathfrak{q}^2)\leqslant\frac{1}{1+\mathfrak{p}^2\mathfrak{q}^2}(\mathfrak{p}\mathfrak{q}).
\end{aligned}\tag{11}$$

Figure 1 depicts the graph that corresponds to inequality (11).

Case 2: Put $\mathfrak{p}=\mathfrak{q}=0.5$, and set the values of $\mathfrak{s}\in[2,2.1]$ and $\mathfrak{t}\in[3,5]$, then (6) becomes

$$\begin{aligned}
&\mathcal{W}\left(\frac{\mathfrak{s}+16\mathfrak{t}}{17}\right)+\frac{12\mathfrak{s}-12\mathfrak{t}}{17}\mathcal{W}'\left(\frac{\mathfrak{s}+16\mathfrak{t}}{17}\right)\leqslant\frac{\int_{\mathfrak{s}}^{\mathfrak{t}}\boldsymbol{u}^2\ {}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}}{\mathfrak{t}-\mathfrak{s}}\leqslant\frac{13\mathfrak{s}^2+4\mathfrak{t}^2}{17}\\
&\left(\frac{\mathfrak{s}+16\mathfrak{t}}{17}\right)^2+\frac{24\mathfrak{s}^2-384\mathfrak{t}^2+360\mathfrak{s}\mathfrak{t}}{(17)^2}\leqslant\frac{2459\mathfrak{s}^2+1400\mathfrak{s}\mathfrak{t}+272\mathfrak{t}^2}{4131}\leqslant\frac{13\mathfrak{s}^2+4\mathfrak{t}^2}{17}\\
&\frac{25\mathfrak{s}^2-128\mathfrak{t}^2+392\mathfrak{s}\mathfrak{t}}{17}\leqslant\frac{2459\mathfrak{s}^2+1400\mathfrak{s}\mathfrak{t}+272\mathfrak{t}^2}{243}\leqslant 13\mathfrak{s}^2+4\mathfrak{t}^2.
\end{aligned}\tag{12}$$

Figure 1 displays the graph corresponding to inequality (12).



(a) Case 1: (b) Case 2:

Figure 1: The graphs that correspond to inequalities (11) and (12) respectively.

Case 3: If we put different values instead of same, i.e. $\mathfrak{p} = 0.2$ and $\mathfrak{q} = 0.5$, then

$$\mathcal{W}\left(\frac{\mathfrak{s}+100\mathfrak{t}}{101}\right)+\frac{90\mathfrak{s}-90\mathfrak{t}}{101}\mathcal{W}'\left(\frac{\mathfrak{s}+100\mathfrak{t}}{101}\right)\leqslant\frac{\int_{\mathfrak{s}}^{\mathfrak{t}}\boldsymbol{u}^2\,{}_{\mathfrak{s}}\widetilde{\mathrm{d}}_{\mathfrak{p},\mathfrak{q}}\boldsymbol{u}}{\mathfrak{t}-\mathfrak{s}}\leqslant\frac{10\mathfrak{t}^2+91\mathfrak{s}^2}{101}$$

$$\left(\frac{\mathfrak{s}+100\mathfrak{t}}{101}\right)^2+\frac{180\mathfrak{s}^2-18000\mathfrak{t}^2+17820\mathfrak{s}\mathfrak{t}}{(101)^2}$$

$$\leqslant\frac{828281\mathfrak{s}^2+181820\mathfrak{s}\mathfrak{t}+10100\mathfrak{t}^2}{1020201}\leqslant\frac{10\mathfrak{t}^2+91\mathfrak{s}^2}{101}$$

$$\frac{181\mathfrak{s}^2-8000\mathfrak{t}^2+18020\mathfrak{s}\mathfrak{t}}{101}\leqslant\frac{828281\mathfrak{s}^2+181820\mathfrak{s}\mathfrak{t}+10100\mathfrak{t}^2}{10101}\leqslant 10\mathfrak{t}^2+91\mathfrak{s}^2. \quad (13)$$

Figure 2 shows the graph associated with the inequality (13).

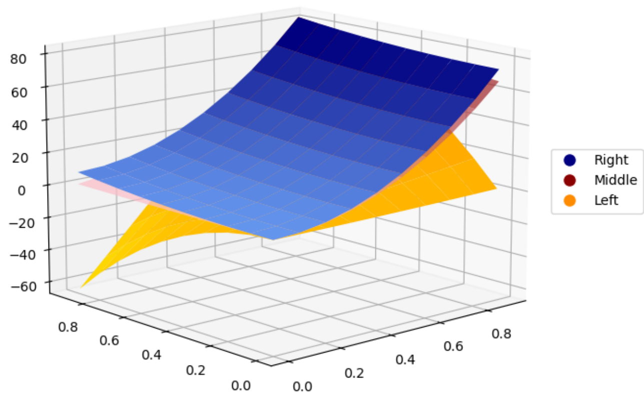


Figure 2: Graphical representation of the inequality (13).

4. Conclusion

The primary features of the left $(\mathfrak{p}, \mathfrak{q})$-symmetric difference operator were initially established in this work. In addition, the fundamental theorem of calculus and the change of variables formula for the left $(\mathfrak{p}, \mathfrak{q})$-symmetric definite integral operators were developed. Furthermore, for convex differentiable functions, two new left $(\mathfrak{p}, \mathfrak{q})$-symmetric Hermite-Hadamard-type inequalities were developed, along with their general form. Finally, the viability of the recently acquired results is studied numerically and graphically. This article might serve as the foundation for future articles in post-quantum calculus, its symmetric version, and inequality theory.

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